Math 116c - Homework 7

Due: May 28, 2008 at 2:30 pm.

This Homework is due Wednesday, May 28 at 2:30 pm. Refer to the grading policy for additional requirements.

1. The goal of the first two exercises is to illustrate how basic "model theoretic" arguments can be used to produce combinatorial results in set theory. Here, we want to show how these techniques would prove that if $\vec{C} = (C_\alpha : \alpha < \omega_1)$ is a sequence of club subsets of ω_1, then $\triangle_{\alpha<\omega_1} C_\alpha$ contains a club.

   (a) Let α be a sufficiently large ordinal such that V_α is a model of enough set theory. (In principle, we could isolate exactly which axioms of ZFC are required to hold in V_α and which formulas need to be absolute for V_α, but in practice this is not really illuminating.) Argue that $\omega_1 \in V_\alpha$ and $V_\alpha \models$ "$x = \omega_1$" iff indeed $x = \omega_1$. Also argue that $\vec{C} \in V_\alpha$ and that for any $C \subseteq \omega_1$, $C \in V_\alpha$ and that C is club iff $V_\alpha \models$"C is club."

   (b) Let $x \in V_\alpha$ be countable. Show that $V_\alpha \models$"x is countable."

   (c) Let $X \prec V_\alpha$ be a countable elementary substructure. Show that $\omega_1 \in X$ and that for any club $C \subseteq \omega_1$, if $C \in X$, then $X \models$"C is club." Show that if f is a function, $y \in \mathrm{dom}\,(f)$, and $f, y \in X$, then $f(y) \in X$. Conclude that if x is countable and $x \in X$ then $x \subseteq X$. In particular, show that $X \cap \omega_1$ must be a limit ordinal, call it γ.

   (d) Show that if $C \subseteq \omega_1$ is club and $C \in X$, then $\gamma \in C$. Conclude that if $\vec{C} \in X$, where $\vec{C}$ is as above, then $\gamma \in \triangle_{\beta<\omega_1} C_\beta$.

   (e) Show that there is a sequence $(X_\beta : \beta < \omega_1)$ of countable elementary substructures of V_α such that all of the following hold:

      i. $\vec{C} \in X_0$.

      ii. If $\beta < \gamma < \omega_1$ then $X_\beta \in X_\gamma$. (Show that this implies that $X_\beta \cap \omega_1 < X_\gamma \cap \omega_1$.)

      iii. If $\beta < \omega_1$ is limit, then $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$.

   Conclude that $\{X_\beta \cap \omega_1 : \beta < \omega_1\}$ is club in ω_1 and deduce from the results above that $\triangle_\xi C_\xi$ contains a club.

2. Given a set X, let $[X]^2 = \{y \subset X : |y| = 2\}$. The classical Ramsey theorem states that if $f : [\omega]^2 \to 2$ then there is an infinite $H \subseteq \omega$ such

that $f\restriction[H]^2$ is constant. It is natural to wonder how this result generalizes to other cardinals.

(a) Show that there is $g : [\omega_1]^2 \to 2$ such that there is no uncountable $H \subseteq \omega_1$ with $g\restriction[H]^2$ constant. (Obviously, there are infinite such H, by the classical Ramsey theorem.)

[`Hint:` Let $X \subseteq \mathbb{R}$ have size $\aleph_1$, and let $\{x_\alpha : \alpha < \omega_1\}$ be an (injective) enumeration of X. For $\alpha < \beta$ define $g(\{\alpha, \beta\}) = 1$ iff $x_\alpha < x_\beta$.]

(b) Nevertheless, the following fact, call it (+), is true:

$f : [\omega_1]^2 \to 2$ then there is a set $H \subseteq \omega_1$ closed in ω_1 and of order type $\omega + 1$ such that $f\restriction[H]^2$ is constant.

Follow the following sketch to provide a proof of (+):

i. Given $f : [\omega_1]^2 \to 2$, let α be a sufficiently large ordinal such that V_α is a model of enough set theory. Let $X \prec V_\alpha$ be a countable elementary substructure such that $f \in X$. Let $\gamma = \omega_1 \cap X$. Suppose that there is no $A \subseteq \omega_1$ cofinal in γ such that $f''[A \cup \{\gamma\}]^2 = \{1\}$. Conclude that there is a finite set $B \subseteq \gamma$ and an ordinal $\delta \in \gamma$ with $\delta > \max(B)$ such that:

A. $f''[B \cup \{\gamma\}]^2 = \{1\}$.

B. For all $\xi \in \gamma$, if $\xi \geq \delta$ then $0 \in f''[B \cup \{\xi, \gamma\}]^2$.

ii. Let $\varphi(\gamma)$ be a formula in the language of set theory (in parameters B, δ, f) expressing facts A,B above, and the fact that $\gamma \geq \delta$. Use that $B, \delta, f \in X$ and the elementarity of X to conclude that $C = \{\tau \in \gamma : \varphi(\tau)\}$ is unbounded in γ.

iii. Show that $f''[C \cup \{\gamma\}]^2 = \{0\}$.

iv. Conclude that (+) indeed holds.

(c) With f, X, etc, as above, use elementarity of X once again to conclude that $\{\beta \in \omega_1 : \varphi(\beta)\}$ is stationary. Therefore, we have shown that either there is a stationary $S \subseteq \omega_1$ such that $f''[S]^2 = \{0\}$, or else there is a closed subset C of ω_1 of order type $\omega + 1$ such that $f''[C]^2 = \{1\}$. Use the result from exercise 2.(a) from Homework 5 to see that for any $\alpha < \omega_1$ either there is a closed $A \subseteq \omega_1$ of order type $\alpha + 1$ such that $f''[A]^2 = \{0\}$ or else there is a C as indicated.

Remark: In fact, Schipperus showed that if $f : [\omega_1]^2 \to 2$ then for any $\alpha < \omega_1$ there is a closed subset $A \subseteq \omega_1$ of order type $\alpha + 1$ such that $f\restriction[A]^2$ is constant. The arguments above are part of an ongoing project with Ramiro de la Vega to reprove Schipperus's result.

3. The argument in exercise 2 above in fact generalizes to any κ regular in place of ω_1. The Erdős-Dushnik-Miller (EDM) theorem says that for any infinite cardinal κ, even if κ is singular, if $f : [\kappa]^2 \to 2$, then either there is $A \subseteq \kappa$ of size κ with $f''[A]^2 = \{0\}$ or else there is an infinite $H \subseteq \kappa$ with $f''[H]^2 = \{1\}$. The following statement, call it $(*)$, is an easy consequence of the EDM theorem:

Given any infinite set X and any two well-orderings $<$ and $\lhd$ of X, there is $Y \subseteq X$, $|Y| = |X|$, such that $< \restriction Y^2 = \lhd \restriction Y^2$.

(a) Show that $(*)$ indeed follows from the EDM theorem.

(b) Provide a direct proof of $(*)$.

[`Hint:` Show that one can reduce to the case $X = \kappa$ an infinite cardinal and $<=\in$. Divide the argument in two cases, according to whether κ is regular (this should be easy) or singular. For the singular case, argue by induction, fix an increasing sequence $(\kappa_\alpha : \alpha < \mathrm{cf}(\kappa))$ cofinal in κ with $\mathrm{cf}(\kappa) < \kappa_0$, and try to build the set Y as $\bigcup_{\alpha<\mathrm{cf}(\kappa)} Y_\alpha$ where $|Y_\alpha| = \kappa_\alpha$ for all $\alpha < \mathrm{cf}(\kappa)$ and if $\alpha < \beta < \mathrm{cf}(\kappa)$ then $\min(Y_\beta) > \sup(Y_\alpha)$ holds both with respect to $\in$ and with respect to $\lhd$.]