Math 116c - Homework 8

Due: June 5, 2008 at 2:30 pm.

This Homework is due at the beginning of lecture on Thursday, June 5. Refer to the grading policy for additional requirements.

The next significant advance in the theory of L after Gödel's work was due to Ronald Jensen in the early 1970s. Jensen introduced the tools for a detailed analysis of L (his "fine structure" theory) and identified several combinatorial statements that hold in L and have proved to be of independent interest. Here we consider one of them.

Definition 1. A $\diamondsuit$-*sequence* is a sequence $(A_\alpha : \alpha < \omega_1)$ such that $A_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$ and whenever $X \subseteq \omega_1$, the set $\{\alpha : A_\alpha = X \cap \alpha\}$ is stationary.

We say that $\diamondsuit$ (*diamond*) holds iff there is a $\diamondsuit$-sequence.

1. Show that $\diamondsuit$ implies CH.

   [Hint: Notice that any $r \subseteq \omega$ is a subset of ω_1, and that $r \cap \alpha = r$ if $\alpha \geq \omega$.]

2. Show that if $\diamondsuit$ holds, then there is a collection of $2^{\aleph_1}$ many stationary subsets of ω_1 such that the intersection of any two of them is countable.

   [Hint: Given $X \subseteq \omega_1$ consider $S_X = \{\alpha : X \cap \alpha = A_\alpha\}$.]

   A natural question is whether the collection $\mathcal{C}$ identified in Exercise 2 is maximal in the sense that given any stationary $T \subseteq \omega_1$, there is some $S \in \mathcal{C}$ with $S \cap T$ stationary (or, at least, unbounded). This was shown in 2003 not to be the case by Cummings and Schimmerling.

3. For $X \subseteq \omega_1$ let $\mathrm{acc}(X) = \{\alpha < \omega_1 : \emptyset \neq \sup(X \cap \alpha) = \alpha\}$ and set $T = \{\alpha < \omega_1 : S_{A_\alpha} \cap \mathrm{acc}(A_\alpha) = \{\alpha\}\}$. Show that T is stationary and $|T \cap S_X| \leq 1$ for any $X \subseteq \omega_1$.

   [Hint: To show that T is stationary, given a club $C \subseteq \omega_1$, consider the least element of $S_C \cap \mathrm{acc}(C)$. For the second part, notice that if $\alpha < \beta$ are members of S_X then $A_\alpha = A_\beta \cap \alpha$.]

   In fact, Cummings and Schimmerling identified ω_2 stationary sets that can be added to $\{S_X : X \subseteq \omega_1\}$ while still mantaining that the intersection of any two of them is bounded.

   Remark: By Exercise 2, a strong negation of $\diamondsuit$ is the statement that in any collection of ω_2 stationary subsets of ω_1, there are at least two with stationary intersection. This statement is known to be consistent with

ZFC and to imply Con(ZFC) and much more. It is still open whether this statement is consistent with CH.

One can generalize $\diamondsuit$ to larger cardinals. Say that $\diamondsuit_{\kappa^+}$ holds iff there is a $\diamondsuit_{\kappa^+}$-sequence, i.e., a sequence $(A_\alpha : \alpha < \kappa^+)$ such that $A_\alpha \subseteq \alpha$ and $\{\alpha : A_\alpha = X \cap \alpha\}$ is stationary in κ^+ for any $X \subseteq \kappa^+$. Exactly as in Exercise 1, $\diamondsuit_{\kappa^+}$ implies that $2^\kappa = \kappa^+$. Jensen showed that CH does not imply diamond. On the other hand, in November last year, Shelah showed that this is the only exception, i.e., that if $\kappa > \omega$ then $2^\kappa = \kappa^+$ implies $\diamondsuit_{\kappa^+}$.

4. Show that $\diamondsuit$ holds in L as follows: If a sequence $(A_\alpha : \alpha < \omega_1)$ is not a $\diamondsuit$-sequence, then there is a set $X \subseteq \omega_1$ and a club $C \subseteq \omega_1$ witnessing this. Jensen built a $\diamondsuit$-sequence in L inductively, by taking care of any such potential counterexample as soon as it appeared: Let $A_\alpha = \emptyset$ unless α is a limit ordinal and there is a set $A \subseteq \alpha$ and a closed unbounded subset C of α such that $C \cap \{\beta < \alpha : A_\beta = A \cap \beta\} = \emptyset$. In that case, let (A, C) be the $<$-least such pair, where $<$ is the well-ordering of L.

   To see that this works, suppose otherwise. Then there is a $<$-least (X, C) witnessing the failure of this sequence. Fix γ limit sufficiently large and let $Y \prec L_\gamma$ be a countable elementary substructure such that $(A_\alpha : \alpha < \omega_1), (X, C) \in Y$. Let $\delta = \omega_1 \cap Y$, let M be the transitive collapse of Y, and let $\pi : Y \to M$ be the collapsing map. Check that $\pi(A_\alpha : \alpha < \omega_1) = (A_\alpha : \alpha < \delta)$, $\pi(X, C) = (X \cap \delta, C \cap \delta)$ and $A_\delta = X \cap \delta$. Derive a contradiction from this situation.