Comments and corrections to '*Super-Real Fields. Totally ordered fields with additional structure*',

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The numbering of the pages follows that of the first edition, Clarendon Press, Oxford, 1996. The numbering of the lines within each page is so that displayed equations are counted in a different line from the one just preceding them.

p. 3 l. 1. Replace "if S is" with "if s is".

p. 6 l. 4. Replace "$\rho < \sigma$" with "$\rho \leq \sigma$". Otherwise, on **l. 8**, $t \restriction \sigma$ may be undefined.

p. 7 ll. -9, -8. Delete "and $T_1 \cup T_2 \neq \emptyset$".

p. 9 l. 4. The proof of (iii) can be simplified: Replace the first 3 paragraphs with:

Let A and B be countable subsets of $\mathbf{Q}$ with $A << B$. Since $A \cup B$ is countable, there exists $\sigma < \omega_1$ such that $\alpha_\tau = 0$ $(\tau \geq \sigma)$ for all $\alpha = (\alpha_\zeta : \zeta < \omega_1)$ in $A \cup B$. Let β be defined as in the proof of (i), i.e., $\beta_\tau = 1$ if there exists a *witness* $\alpha \in A$ with $\alpha_\tau = 1$ and $\alpha \restriction \tau = \beta \restriction \tau$, and $\beta_\tau = 0$ otherwise. Set $\gamma_\sigma = 1$ and $\gamma_\tau = \beta_\tau$ $(\tau \neq \sigma)$. Clearly, $\gamma \in \mathbf{Q}$ and $A << \gamma$. Notice that if $\tau < \omega_1$, $\beta_\tau = 1$ and $\alpha \in A$ is a witness, then $\delta(\alpha, \gamma) > \tau$.

In the last paragraph, replace each mention of β with η, and "By the claim, [...] $\delta(\alpha, \gamma) > \tau$." with "Take $\alpha \in A$ with $\delta(\alpha, \gamma) > \tau$." Similarly, replace "By the claim again, [...] $\delta(\alpha, \gamma) > \rho$." with "Take $\alpha \in A$ with $\delta(\alpha, \gamma) > \rho$."

If these replacements do not take place, notice the typo on **l. 13**: "Necessarily $\tau < \sigma$" should read "Necessarily $\tau \leq \sigma$". It is easy to check that γ as defined in the book, and as defined here coincide. But the inductive definition is easier, and had already been used anyway.

p. 10 l. -1 and p. 1 1 l. 1. The $(\aleph_1, \aleph_1)$-pregap described in the text is not a gap. This description is not correct, anyway. Taken literally, the sequences indicated are just $\delta^{(\sigma)}$ and $\alpha^{(\sigma)}$ themselves, although it seems clear that the intention was to define two sequences which $\delta^{(0)}$ interpolates. Woodin indicated 2 possible corrections, and a proof from CH:

(1) First, the proof under CH (Compare with the proof of Proposition 1.17): Let $\hat{\mathbf{S}} = \{0,1\}^{<\omega_1}$. To each $\alpha \in \hat{\mathbf{S}}$ we assign an interval I_α inside $\mathbf{Q}$ as follows: $I_{\langle\rangle} = \mathbf{Q}$. Given $\alpha \in \hat{\mathbf{S}}$ such that $I_\alpha \neq \emptyset$, pick $p \in I_\alpha$ and set $I_{\alpha^\frown 0} = I_\alpha \cap (-\infty, p)$ and $I_{\alpha^\frown 1} = I_\alpha \cap (p, \infty)$. For $\alpha \in \hat{\mathbf{S}}$ with lenght $l(\alpha)$ a limit ordinal such that I_β is defined for each β a proper initial segment of α, set $I_\alpha = \cap_{\rho < l(\alpha)} I_{\alpha \restriction \rho}$. An easy inductive argument using that $\mathbf{Q}$ is an η_1-set shows that I_α is defined and nonempty for each $\alpha \in \hat{\mathbf{S}}$.

For $f \in \mathbf{S} = \{0,1\}^{\omega_1}$ let $I_f = \cap_{\rho<\omega_1} I_{f\restriction\rho}$. Suppose I_f is empty. Let

$$\begin{aligned} A_f &= \{\, \alpha \in \mathbf{Q} : \text{ for some } \tau < \omega_1,\ \alpha << I_{f\restriction\tau} \,\}, \\ B_f &= \mathbf{Q} \setminus A_f. \end{aligned}$$

Then $\langle A_f, B_f \rangle$ is a gap in $\mathbf{Q}$. Notice that at most $\mathfrak{c}$ many of these gaps are $(0, \aleph_1)$, $(\aleph_1, 0)$, $(\aleph_0, \aleph_1)$ or $(\aleph_1, \aleph_0)$-gaps. Also notice that for at most $|\mathbf{Q}| = \mathfrak{c}$ many functions f it is the case that $I_f \neq \emptyset$. Since CH holds, $\mathfrak{c} < 2^{\aleph_1} = |\{0,1\}^{\omega_1}|$, so there is at least one f such that $I_f = \emptyset$ and $\langle A_f, B_f \rangle$ is an $(\aleph_1, \aleph_1)$-gap in $\mathbf{Q}$.

(2) A general construction, which will be useful elsewhere: Recall that $\mathbf{Q} = \bigcup \{\, \mathbf{Q}_\sigma : \sigma < \omega_1 \,\}$ is the union of an increasing chain of α_1-sets. We use the following:

Claim 1 *If λ is limit, for any $\alpha < \gamma \in \mathbf{Q}_\lambda$ there is a gap $\langle A, B \rangle$ in $\mathbf{Q}_\lambda$ and a nontrivial interval $I \subset \mathbf{Q}$ such that $\alpha << A << I << B << \gamma$.*

Dem. Define inductively a strictly increasing sequence $(\, \alpha_\tau : \tau < \rho \,)$ and a strictly decreasing sequence $(\, \gamma_\tau : \tau < \rho \,)$ of elements of $\mathbf{Q}_\lambda$ such that

1. $\alpha < \alpha_0$, $\gamma_0 < \gamma$;
2. whenever $\tau_1, \tau_2 < \rho$, $\alpha_{\tau_1} < \gamma_{\tau_2}$;
3. the sequences $(\, \sigma_\tau : \tau < \rho \,)$ and $(\, \eta_\tau : \tau < \rho \,)$, where σ_τ (respectively, η_τ) is the last ordinal μ such that $\alpha_\mu = 1$ (respectively, $\gamma_\mu = 1$), are strictly increasing.

Keep the construction until a stage is reached where it cannot be continued. It is clear that this stage ρ is a limit ordinal, and that $\rho \leq \lambda$. Let $A = \{\, \alpha_\tau : \tau < \rho \,\}$, $B = \{\, \gamma_\tau : \tau < \rho \,\}$. Define β as in the proof of Proposition 1.9(i). Set $\beta^1_\delta = \beta_\delta$ $(\delta \neq \lambda)$, $\beta^1_\lambda = 1$, and $\beta^2_\delta = \beta_\delta$ $(\delta \neq \lambda, \lambda + 1)$, $\beta^2_\lambda = \beta^2_{\lambda+1} = 1$. Then $A << (\beta^1, \beta^2) << B$. □

An easy induction using the claim gives a sequence of nested intervals $(\, I_\tau : \tau < \omega_1 \,)$ such that $I_\tau \cap \mathbf{Q}_{\omega\tau} = \emptyset$ $(\tau < \omega_1)$. Set

$$\begin{aligned} A &= \{\, \alpha \in \mathbf{Q} : \text{ for some } \tau < \omega_1,\ \alpha << I_\tau \,\}, \\ B &= \mathbf{Q} \setminus A. \end{aligned}$$

Then $\langle A, B \rangle$ is an $(\aleph_1, \aleph_1)$-gap in $\mathbf{Q}$, for if $A << f << B$, then $f \in \mathbf{Q} \setminus \bigcup\{\, \mathbf{Q}_{\omega\tau} : \tau < \omega_1 \,\}$, a contradiction.

(3) A specific example: Let $\alpha = (1010\ldots)$, or more generally, any element of $\mathbf{S}$ which cofinally often takes the value 0 and the value 1. Then $\langle(-\infty, \alpha) \cap \mathbf{Q}, (\alpha, \infty) \cap \mathbf{Q}\rangle$ defines a gap in $\mathbf{Q}$. It is an $(\aleph_1, \aleph_1)$-gap, because the supremum (or infimum) of any countable sequence in $\mathbf{Q}$ takes eventually the value 0.

p. 15 l. 5. Replace "at most" with "strictly less than".

p. 15 ll. 14, 15. Replace each appearance of S_0, S_1 with I_0, I_1, respectively.

p. 15 l. -12. Replace "$\subsetneq$" with "$\supsetneq$".

p. 15 l. -11. Add "Let $\tilde{T} = \{\, a : a{\restriction}\sigma \in T \,(\sigma < l(a))\, \}$." Replace "$T$" with "$\tilde{T}$".

p. 15 l. -8. Replace "T" with "$\tilde{T}$".

p. 16 l. 17. Add a reference (the Notes at the end of the chapter do not talk about this point), e.g: "See Fuchs (1963) for a proof of these equalities."

Actually, the only one which is not immediate is $|x| = x^+ - x^-$. It is easy to see that this follows once $|x| \geq 0$ is established. But $2|x| \geq 0$. Now: $2(|x| \wedge 0) = 2|x| \wedge |x| \wedge 0 = |x| \wedge 0$, so $|x| \wedge 0 = 0$, and $|x| \geq 0$, as wanted.

p. 17 l. -5. Replace "$(x, y \in G)$" with "$(x, y \in G \setminus \{0\}, x + y \neq 0)$".

p. 17 l. -3. Replace "$x, y \in G^+$" with "$x, y \in G^+ \setminus \{0\}$".

p. 19 ll. -11, -10. This is wrong. A countable, well-ordered subset of $\mathbb{R}$ does not necessarily have "the form $\{\, s_n : n \in \mathbb{N} \,\}$ for an increasing sequence (s_n)"; the set $\{\, -\frac{1}{n} : n \in \mathbb{N} \,\} \cup \{0\}$ is a counterexample. In fact, any countable ordinal $\alpha < \omega_1$ can be the order type of a well-ordered subset of $\mathbb{R}$ (and any such subset must be countable).

p. 20 l. 13. This is false as stated. It is true if $\mathfrak{F}_{(1)}$ is replaced with $\mathfrak{F}_{(0)}$, or if $\kappa^{\aleph_0} = \kappa$; under CH, this last condition holds whenever $\aleph_0 < \kappa < \aleph_\omega$.

p. 20 l. 15, 16. Replace "κ" with "$\kappa \geq \omega$". I only mention this because it is used in the proof of Theorem 5.9, which requires extra-hypothesis to hold.

p. 20 l. 18. The mistake in the proof occurs in the equality $\kappa^{\aleph_0} = \kappa$. This is not necessarily true, and $\kappa = \aleph_\omega$ is a counterexample. Thus, the proposition is false, for example, if $\mathfrak{c} < \aleph_\omega$ and $S = \omega_\omega$.

p. 23 l. 5. Add "Without loss of generality, $(x, 1) > 0$."

p. 23 ll. -15, -14. Replace "$f \in$" with "$f - h_\sigma \in$", and "$g \in$" with "$g - h_\sigma \in$".

p. 30. The proof of Proposition 1.41 is incomplete in an essential way. The problem appears quite early, on **l. 3**, where the map $\overline{\psi}$ is defined. It is not clear that $\overline{\psi}$ exists, since it is not clear that ψ must be 'completely' isotonic (in the obvious sense).

On **l. 8** it seems that the condition $f_\tau \leq f_\sigma$ should be added, to ensure that I'_σ, as defined on **l. 12**, is nonempty. Even if this follows from the other assumptions (I am not sure it does), its insertion as an extra clause is harmless, and actually clarifies what the construction intends to accomplish.

On **l. 15**, "by f_σ." must be replaced with "by $\psi(f_\sigma)$."

On **l. 18**, the set $\{\, v(f) : f \in A \cup B \,\}$ is simply $v(G_\sigma)$.

On **l. -10**, "no isotonic" must be replaced with " no completely isotonic".

The following proof of the proposition, as stated, was provided by Woodin:

Since $\operatorname{cof} S \geq \aleph_1$, it is easy to see that there is an isotonic map from $\mathbf{S} = \{0, 1\}^{\omega_1}$ into $\mathfrak{F}(\mathbb{R}, S)^+$. By Theorem 1.39(iv), $\mathfrak{F}_{(1)}(\mathbb{R}, T)$ is a β_1-set, and there is an isotonic map from it into $\mathbf{Q}$, by Theorem 1.13. Thus, in order to prove the proposition, it suffices to show that there is no isotonic map from $\mathbf{S}$ into $\mathbf{Q}$.

Toward a contradiction, suppose there is one. Let $\mathbf{Q}^*$ be the subset of $\mathbf{S}$ obtained by adding to $\mathbf{Q}$ all the functions filling $(\aleph_1, \aleph_1)$-gaps of $\mathbf{Q}$. It is easy to see that $\mathbf{Q}^*$ itself does not have such kind of gaps. Write $\mathbf{Q}^* = \bigcup\{\, \mathbf{Q}^*_\sigma : \sigma < \omega_1 \,\}$ as a union of a chain of α_1-sets; this is possible because $\mathbf{Q}^*$ embeds into $\mathbf{Q}$. Now a contradiction is easy to obtain: Exactly as in the second of the proofs of Proposition 1.11(ii) shown above, two sequences $(a_\sigma : \sigma < \omega_1)$ and $(b_\sigma : \sigma < \omega_1)$ of elements of $\mathbf{Q}^*$ can be defined such that, letting

$$I_\sigma = (a_\sigma, b_\sigma) \quad (\sigma < \omega_1),$$

$(\, I_\sigma : \sigma < \omega_1 \,)$ is a nested sequence of intervals and $I_\sigma \cap \mathbf{Q}^*_\sigma = \emptyset$ $(\sigma < \omega_1)$. Let $f \in \mathbf{Q}^*$ interpolate the gap $\langle \{\, a_\sigma : \sigma < \omega_1 \,\}, \{\, b_\sigma : \sigma < \omega_1 \,\} \rangle$. Then $f \in \mathbf{Q}^* \setminus \bigcup\{\, \mathbf{Q}^*_\sigma : \sigma < \omega_1 \,\} = \emptyset$, a contradiction.

p. 32 l. 1. Replace "semi-η_1-group" with "divisible semi-η_1-group".

p. 33 l. -10. Replace "$G = (\aleph_0, 1)$" with "$\delta(G) = (\aleph_0, 1)$".

p. 39 l. -9. Replace “$a \in \mathrm{Inv}K$” with “$a \in \mathrm{Inv}K^{\#}$”.

p. 43 l. 10. Replace “$\mathbb{K}$” with “$\mathbb{F}$”.

p. 43 l. 11. Replace “$\alpha a_1 b_1$” with “$\alpha a_2 b_1$”.

p. 44 l. -1. Since the elements of B have not been realized as functions, it does not make sense to claim that “But, for each $x \in S$, necessarily $f(x) = b(x)$, and so $S << w(b - f)$.” This can be easily fixed, though; just replace this sentence with:

For $x \in S$, fix $a \in A$ with $x = w(b - a)$. If $u = w(f - a) < x$, then $f(u) = a(u)$ (by definition of f), contradicting the definition of u. So $w(b - f) \geq \min\big(w(b - a), w(f - a)\big) = x$. Hence, $S << w(b - f)$.

p. 47 ll. 10–18. Proposition 2.6 does not apply directly, since it is a result on real, not complex, fields. But the argument shown can be slightly modified to give the wanted result: Let $C = B \cap R$, where R is the real closure of $\mathfrak{F}$ inside the quotient field of B. It is easy to check that C is a valuation algebra. The argument in the book now applies to $\mathfrak{F}, C$ instead of A, B, giving that $\mathfrak{F}$ is real-closed.

p. 48 l. -8. The statement is slightly inaccurate, since $p(a) = 0$ and $p(b) = 0$ are possible (if, say A has a maximum or B a minimum).

p. 50 l. -14. Replace “Then the” with “Then, in particular,”.

p. 51 l. -5. Replace “$\mathbf{R} \times \mathbf{G} \times \mathbb{R}$” with “$\mathbb{R} \times \mathbf{G} \times \mathbb{R}$”.

p. 52 l. 1. The wording is a bit loose, since, if ‘interval’ is just intended to mean ‘(open) convex’ (as defined on **p. 5**), the statement may be false. To fix it, it suffices to replace “interval in Γ_K” with “interval $(s, t) \in \Gamma_K$ (for some $s, t \in \Gamma$)”.

p. 52 ll. 3–5. Add “(At least under CH.) See also Corollary 2.35”.

p. 53 l. -11. Replace “$\aleph_2$” with “$\aleph_2 \cdot \mathfrak{c}$”. Notice that if $\mathfrak{c} > \aleph_2$, the proposition as stated is trivially false.

p. 53 l. -1. Replace “$|K| = |G| = |T| = \aleph_2$.” with “$|K| = |G| = |T| = \aleph_2$, if $\aleph_2 \geq \mathfrak{c}$. Otherwise, obviously $|K| = \mathfrak{c}$.” The mistake lies not in this Proposition itself, but in Proposition 1.28(i), where, as remarked above, extra-hypothesis are required to warrant the result.

p. 54 l. 6. Replace “Theorem 2.27” with “Theorem 2.17”.

p. 54 ll. 12, 13. Replace both ocurrences of “s” with “$\hat{s}$”, or something similar. Recall that s was already introduced (in the previous paragraph) as $v(a)$.

p. 55 l. -16. Replace "$\mathbb{R}$" with "$\mathbf{R}$".

p. 56 l. -4. Replace "$\rho \le \mathfrak{c}$" with "$|\rho| \le \mathfrak{c}$". *A priori*, any $\rho < \mathfrak{c}^+$ can be realized as the length of such a chain.

p. 58 ll. 1–4. This is a strange remark, because obviously in the absence of CH, there are no real-closed η_1-fields of size $\aleph_1$.

p. 58 l. -2. Replace "29)" with "29".

p. 64 l. 4. Replace "$\mathbf{S}_\sigma$" with "$\{0,1\}^\sigma$".

p. 67 ll. 7–13. This argument must be modified since, as already pointed out, the proof of Proposition 1.41 given in the book is essentially incomplete. The alternative argument shown here can be used instead:

Let $(\, s_\sigma : \sigma < \omega_1 \,)$ be a strictly increasing sequence of elements of $\mathbf{G}$, and define $\psi : \mathbf{S} \to \tilde{\mathbf{R}}^+$ by setting for $a = (\, a_\sigma : \sigma < \omega_1 \,) \in \mathbf{S}$, $\psi(a)$ to be the function with support contained in $\{\, s_\sigma : \sigma < \omega_1 \,\}$ such that $\psi(a)(s_\sigma) = a_\sigma$ $(\sigma < \omega_1)$. Notice $\psi(a) \in \tilde{\mathbf{R}}^+$ since it belongs locally to $\mathbf{R}^+$. Then ψ is isotonic, and if there is an isotonic map from $\tilde{\mathbf{R}}^+$ into $\mathbf{R}$, then there is one from $\mathbf{S}$ into $\mathbf{Q}$ (because $\mathbf{Q}$ and $\mathbf{R}$ are isomorphic as ordered sets). But this is impossible, by the (new) proof of Proposition 1.41.

p. 70 l. 16. Replace "is isomorphic to" with " is isomorphic (as an algebra) to".

p. 75 l. -1. Replace "$\pi_P(f')$" with "$\pi_P(f)$".

p. 78 l. -15. Replace (twice) "$C(\Phi_A)$" with "$C(\Phi_A, \mathbb{C})$".

p. 79 l. 7. Maybe "character" should be replaced with "real character". Formally, a character is a non-zero complex algebra homomorphism with range $\mathbb{C}$. (In Chapter 2 both real- and complex-valued characters where defined, but in page 78 only the complex case is mentioned.)

p. 80 l. 17. Replace "4.20(iv)" with "4.20(v)".

p. 80 l. 19. Delete "so, by (iii),", since the statement that follows is actually part of (v).

p. 80 l. -8. This is false. The following is a counterexample:

Let $\Omega = \mathbb{N} \cup \{\infty\}$, so $C(\Omega) = c$ in the notation introduced on page 69. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and set

$$P = \{\, f \in c : \mathbf{Z}_{\mathbb{N}}(f|\mathbb{N}) \in \mathcal{U} \,\}.$$

Then P is prime and non-maximal. Given $f, g \in c$, $\pi_P(f) = \pi_P(g)$ iff $f(\infty) = g(\infty)$ and $f \sim_{\mathcal{U}} g$. Hence,

$$c/P \cong \mathbb{R} \times (c_0/\mathcal{U}),$$

where $c_0/\mathcal{U}$ is the image of c_0 inside $\mathbb{R}^{\mathbb{N}}/\mathcal{U}$. Thus, $M_P = (c_0/\mathcal{U}) \cap K_P^0 = c_0/\mathcal{U}$ (more exactly, its image inside c/P).

We *claim* that if $c_0/\mathcal{U}$ is cofinal in K_P^0, then $\mathcal{U}$ is a P-point (i.e., $\mathfrak{p}$ is, where $\mathfrak{p} \in \beta\mathbb{N} \setminus \mathbb{N}$ is the point corresponding to $\mathcal{U}$.). In effect, suppose that whenever $0 \leq f_1, f_2 \in c$ and $\pi_P(f_1)/\pi_P(f_2) \in K_P^0$, there is $f \in c_0$ such that

$$\frac{\pi_P(f_1)}{\pi_P(f_2)} \leq [f]_{\mathcal{U}}.$$

Fix such f_1, f_2 and f. For each $n \in \mathbb{N}$, let

$$A_n = \{\, m : f_1(m) \leq f_2(m)/n \,\}.$$

Then $A_n \in \mathcal{U} \quad (n \in \mathbb{N})$. Also, letting

$$A = \{\, m : f_1(m) \leq f_2(m)f(m) \,\},$$

$A \in \mathcal{U}$. But $f(m) \to 0$ as $m \to \infty$, so $A \setminus A_n$ is finite $(n \in \mathbb{N})$.

Recall that $\mathfrak{p}$ is a P-point iff every G_δ subset of $\beta\mathbb{N} \setminus \mathbb{N}$ containing $\mathfrak{p}$ is a neighbourhood (in $\beta\mathbb{N} \setminus \mathbb{N}$) of $\mathfrak{p}$. In Theorem 2.24 of Dales and Woodin (1987) it is shown that this is equivalent to the statement that whenever $\{\, B_n : n \in \mathbb{N} \,\} \subset \mathcal{U}$, there is $B \in \mathcal{U}$ such that $B \setminus B_n$ is finite $(n \in \mathbb{N})$. Suppose that $\mathcal{U}$ is not a P-point, and fix a 'witness' $\{\, A_n : n \in \mathbb{N} \,\}$. By replacing each A_n with $A_n' = \bigcap_{m \leq n} A_m$, we may assume

$$A_1 \supset A_2 \supset \cdots .$$

By thinning down the sequence, we may further assume that each containment is proper. But now it is easy to define functions $f_1, f_2 \in c^+$ such that $f_1(m) \leq f_2(m)/n$ iff $m \in A_n$ $(n \in \mathbb{N})$, and the argument of the previous paragraph shows that $c_0/\mathcal{U} << \pi_P(f_1)/\pi_P(f_2) \in K_P^0$. This proves the claim.

Finally, we claim that there is a free ultrafilter $\mathcal{U}$ which is not a P-point. For this, fix a partition $\{\, A_n : n \in \mathbb{N} \,\}$ of $\mathbb{N}$ into infinite sets, and let $\mathcal{F} = \{\, X \subset \mathbb{N} : \text{ for all but finitely many } n,\ A_n \setminus X \text{ is finite} \,\}$. Then $\mathcal{F}$ is a filter, and no ultrafilter $\mathcal{U}$ extending $\mathcal{F}$ is a P-point: Notice that $\mathbb{N} \setminus A_n \in \mathcal{F}$ $(n \in \mathbb{N})$. Suppose that $X \subset \mathbb{N}$ is such that $X \setminus (\mathbb{N} \setminus A_n) = X \cap A_n$ is finite $(n \in \mathbb{N})$. Let $Y = \mathbb{N} \setminus X$. Then $A_n \setminus Y = A_n \cap X$ $(n \in \mathbb{N})$, so $Y \in \mathcal{F}$. Thus, $X \notin \mathcal{U}$, and $\mathcal{U}$ is not a P-point, completing the proof of the claim.

Hence, letting $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ which is not a P-point, Ω and P as defined above contradict Proposition 4.21(iii).

(An approach more topological in nature is possible. As a matter of fact, this example is discussed in the book, see after Proposition 4.39. The construction of the filter $\mathcal{F}$ follows Jech (1978), but also see the proof of Theorem 4.47.)

p. 81. The proof of Proposition 4.21(iii) is incorrect, of course. The following inaccuracies and errors where detected:

On **l. 4**, "K_P^0" must be replaced with "${K_P^0}^+$".

On **l. 6**, it is false in general that each F_n is a neighbourhood of x_P. The following is a counterexample: With $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, let Ω and P be as above, and fix a coinfinite set $A \in \mathcal{U}$. Set $f(\infty) = g(\infty) = 0$ and

$$f(n) = \begin{cases} \frac{1}{n} & n \notin A, \\ \frac{1}{n^2} & n \in A, \end{cases} \quad \text{and} \quad g(n) = \begin{cases} \frac{1}{n^2} & n \notin A, \\ \frac{1}{n} & n \in A. \end{cases}$$

Then $f, g \in M_\infty^+$ and $\pi_P(f)/\pi_P(g) \in {K_P^0}^+$, but no F_n is a neighbourhood of $x_P = \infty$, for there are arbitrarily large values of $m \in \mathbb{N}$ for which

$$(n+1)f(m) > g(m)$$

(The set of $m \notin A$ with this property is in $\mathcal{U}$, as a matter of fact).

On **l. 7**, h is not necessarily well defined, although this is not essential, as it would suffice to replace the definition given in the book with

$$h(x) = \inf\{ \frac{1}{n} : x \in \partial F_n \} \quad (x \in \Omega).$$

Then, replace $(F_n : n \in \mathbb{N})$ with $(F_{n_i} : i \in \mathbb{N})$, where $F_{n_1} \supsetneq F_{n_2} \supsetneq \dots$ and, letting $n_0 = 0$, $F_k = F_{n_i} \quad (n_{i-1} < k \le n_i,\ i \in \mathbb{N})$.

Finally, on **l. 10**, it is not clear that "$f \le hg$ in $C(\Omega)$", although for sure it holds in F_1. But notice that since the F_n are not necessarily neighbourhoods, the desired continuous extension h may fail to exist.

p. 82 l. 5. Replace the first appearance of "$C^b(X)$" with "$C^b(X, \mathbb{C})$".

p. 83 l. -9. The hypothesis that $K_P \ncong \mathbb{R}$ is missing.

p. 85 l. 2. Replace "$\delta_x \le p_x(w)$" with "$\delta_x \le |p_x(w)|$".

p. 85 l. -9. In (iii), add "(If $K_P \ncong \mathbb{R}$ and M_P is cofinal in K_P^0)". At the moment I do not know if the result is true without extra assumptions on P or K_P.

p. 87 ll. 9–14. Since Proposition 4.21(iii) is false in general, what this argument shows is that $\operatorname{cof} M_p \ge \aleph_1$, so $\delta(\Gamma_P) \ge \aleph_1$ if M_P is cofinal in K_P^0, e.g., if P is a valuation prime.

p. 87 l. -7. Add "and $g > 0$".

p. 88 l. -13. Add a reference, like: "See Gillman and Jerison (1960, 13.2)".

p. 89 l. -10. Add "(See the remarks at the end of the Notes, page 107.)" It seems easier just to argue directly: Since c_0 is an ideal (in c), c_0/P is an ideal in K_P.

p. 89 l. -5. Add "(Without loss of generality, d exists)".

p. 91 l. -9. Replace "$\mathcal{U}$" with "$\mathcal{V}$".

p. 91 l. -6. This is false in general. It is easier to check, and suffices here, that $A_Q \cong \big(l^\infty \times (c_0/\mathcal{V})^{\mathbb{N}}\big)/\mathcal{U}$.

p. 91 l. -5. It would be nice to expand "$\delta(K_Q) = \delta(K_P) \geq \aleph_1$" into "$\delta(K_Q) = \delta(A_Q) = \delta(l^\infty/\mathcal{U}) = \delta(K_P) \geq \aleph_1$, by Proposition 4.21(ii)".

p. 93 l. 2. Add "$(r_0 = \infty)$".

p. 93 l. -15. Replace "$(1/n)(e_n \circ f)$" with "$(1/n)(e_n \circ f)(x)$".

p. 93 l. -14. Replace "φ" with "$\varphi = x_P$".

p. 93 l. -11. I do not see an argument for the non-modularity of Q any simpler than the following; since it appeals to the Notes at the end of the chapter, maybe it should be included:

As in the Notes, page 107, let $\hat{P} = \{\, f \in l^\infty : f[c_0] \subset Q \,\}$. Then $\hat{P}$ is a prime ideal in l^∞ and $\hat{P} \cap c = Q$. If $\hat{h} \in c_0$ is such that $\hat{h} + Q$ is the identity in c_0/Q, $\hat{h} \notin \hat{P}$ and $\hat{h} + \hat{P}$ is an identity in $l^\infty/\hat{P}$. Since $(1) + \hat{P}$ is also an identity, and $\mathbf{Z}\big(\hat{h} - (1)\big) \subset \mathbb{N}$ (looking at $\hat{h}$, (1) as functions in $C(\beta\mathbb{N})$), there is $n \in \mathbb{N}$ such that

$$h(\hat{P}) = \bigcap \{\, \mathbf{Z}(f) : f \in \hat{P} \,\} = \{n\}.$$

This is impossible, as $c_{00} \subset \hat{P}$.

p. 94 l. -14. Replace "$\mathbb{Z}^+$" with "$\mathbb{N}$".

p. 95 l. 17. Replace "$P_\sigma \subsetneqq P_\tau$" with "$P'_\sigma \subsetneqq P'_\tau$".

p. 97 l. -6. Replace "Q" with "$J_{\mathfrak{P}}$".

p. 99 ll. 12–14. Better, set $x_n = 2/3^n$, $y_n = 1/3^{n-1}$, so $1 = y_1$ is included in the union $\bigcup\{\, [x_n, y_n] : n \in \mathbb{N} \,\}$ (notice $1 \in K$). This makes the strict inequality $y_n < 1/n$ false for $n = 1$, but it does not matter for the construction. The same change is necessary at the other stages of the definition of the families $\mathcal{I}_k$.

p. 99 l. -9. Replace "$1/n$" with "$x + 1/n$".

p. 101 l. 10. Replace "$\mathbf{Z}(g) = \mathbf{Z}(h)$" with "$\mathbf{Z}(g) \cap W = \mathbf{Z}(h) \cap W$" (since it is possible that $\mathbf{Z}(h) \not\subset W$).

p. 102 l. 8. Replace "$\{ m \in \mathbb{N}^k : I_m \cap W \neq \emptyset \}$ belongs to $\mathcal{U}_k$" with "$\{ m \in \mathbb{N}^{k+1} : I_m \cap W \neq \emptyset \}$ belongs to $\mathcal{U}_{k+1}$".

p. 102 l. -16. Add "We may assume that $\overline{V_1} \cap \overline{V_2} = \emptyset$".

p. 102 ll. -7, -6. Replace each appearance of "$V_1 \cap V_2$" with "$\overline{V_1} \cap \overline{V_2}$".

p. 104. Replace both black arrowheads with white arrowheads. See the next remark.

p. 105 ll. -22--18. Replace "4.11" with "5.11".

Notice this theorem requires GCH. It would be good to expand on this remark: Theorem 5.11 of Antonovskij *et al.* (1981) shows (in ZFC) that there is a hyper-real field L with $|L| = \beth_{\omega_1}$. It also quotes results that, under GCH, guarantee that any ultrapower of $\mathbb{R}$ has size κ, where either $\kappa = \mathfrak{c}$ (which in this case is $\aleph_1$, because of CH, but not otherwise, as stated on **l. -19**), or $\kappa = \kappa^{\aleph_1}$. 5.11 is *not* a result in ZFC, so it cannot be used to justify Fig. 4.1 on **p. 104**.

This being the case, it cannot be assumed any longer that "A similar argument shows that there is a field in **cut prime z-ideal η_1** which is not an ultrapower", as stated on **l. -18**.

I do not know if the results, as stated, are theorems in ZFC. In fact, it is still open whether an ultrapower of $\mathbb{R}$ can be of size $\beth_{\omega_1}$.

p. 110 l. 9. After "$z \neq 0$," add "($z \in \mathbb{C}$)".

p. 110 l. -15. After "Theorem 1.28(ii)" add "and Hahn's embedding theorem".

p. 110 l. -8. Recall that Proposition 1.28(i) is false. It applies in this case, though, because $\aleph_2^{\aleph_0} = \aleph_2$ under CH.

p. 110 l. -2. Replace "finite" with "finite non-empty".

p. 111 l. 2. Replace both instances of "$\cup$" with "$\cap$".

p. 112 ll. -18, -17. After *"ordinal."* add *"Let σ^* be σ with the reverse ordering, i.e., for $\alpha, \beta < \sigma$, set $\alpha <_{\sigma^*} \beta$ if and only if $\beta < \alpha$. Let $\hat{\sigma} = \sigma^* \odot \sigma$ (see page 3)"*.

Replace "$\mathfrak{F}_{(0)}(\mathbb{Q}, \sigma) = \{ u \in \mathbb{Q}^\sigma : \ldots$" with "$\mathfrak{F}_{(0)}(\mathbb{Q}, \hat{\sigma}) = \{ u \in \mathbb{Q}^{\hat{\sigma}} : \ldots$".

Without this modification, Theorem 5.9 is false; the computation of $\delta(L_\sigma)$ may fail.

p. 112 l. -14. Replace *"ordinal"* with *"ordinal such that* $\operatorname{cf}(\sigma) = |\sigma|$*"*. It is not true (even with our modification) that $\delta(L_\sigma) = |\sigma|$, although $\delta(L_\sigma) = \operatorname{cf}(\sigma)$ holds, provided that $\operatorname{cf} \sigma$ is infinite (i.e., that σ is a limit ordinal). Notice that the proof invokes Proposition 1.28, but the results there only hold (in the notation of that Proposition) for $\kappa \geq \omega$.

p. 112 l. -9. Replace "2^κ" with "2^κ, exactly as in the proof of Proposition 1.9(iv)".

p. 115 l. 3. Replace "$[m_{i+1}, m_{i+1}]$" with "$[m_i + 1, m_{i+1}]$".

p. 115 l. -5. Replace "$(s^{(1)}, \dots, s^{(n)})$" with

$$(\tilde{s}^{(1)}, \dots, \tilde{s}^{(n)}), \text{ where } \tilde{s}^{(k)} = s^{(k)} | [1, m_i] \quad (k \in \{1, \dots, n\}).$$

p. 117 l. -14. After *"semi-algebraic"* add *"continuous"*.

p. 118 l. 7. Replace "$x \in B_n(0; \varepsilon)$, and $B_n(0; \varepsilon)$" with "$x \in S_n(0; \varepsilon)$, and $B_n(x; \delta)$".

p. 118 ll. -18, -17. Since Łojasiewicz inequality was stated without all the hypotheses required for it to hold, it must be checked that this is the case:

Fix ε_1 with $0 < \varepsilon_1 < 1, \varepsilon_0$. It is easy to check that $h|(0, \varepsilon_1]$ is continuous. Since $0 \leq h \leq 1$, by Benedetti and Risler (1990, Lemma 2.3.12), $\lim_{x \to 0^+} h(x) = a$ exists. By the Tarski-Seidenberg theorem, $\{(0, a)\} \cup h|(0, \varepsilon_1]$ is semi-algebraic. Now Theorem 5.20 applies (with ε_1 instead of ε_0).

p. 119 l. 3. Replace "V" with "U".

p. 120 l. 6. Replace "U" with "U_p".

p. 120 l. 15. Add "$= F_{(p^\frown q; b^\frown c)}$".

p. 120 l. 18. Add "(By Proposition 4.36)".

p. 121 l. -17. After "$\dots, a_n\}$" add "$, 0 < \delta < \varepsilon$".

p. 122 l. -17. Replace "$(\mathbb{R}^\kappa / \mathcal{V})^0$" with "$\mathbb{R}^\kappa / \mathcal{V}$".

p. 123 l. 2. Replace "$\mathbb{C}((X))$" with "$\mathbb{C}(X)$".

p. 124 ll. 1–5. Add CH as a general hypothesis of the theorem, and remove it from just statement (ii). It is not clear that 5.28(i) may fail without CH, but the proof in the book does.

p. 124 l. 7. Replace "$c_0 \subset Q$" with "$c_{00} \subset Q$".

p. 124 l. 14. The fact that $|\Gamma_Q| = \aleph_1$ may fail without CH. The following is a counterexample:

If MA + ¬CH holds, and P is the maximal ideal of l^∞ above

$$\hat{Q} = \{\, f : f[c_0] \subset Q \,\},$$

then $\operatorname{coi}\left((c_0/P)^+ \setminus \{0\}\right) = \mathfrak{c} > \aleph_1$. Hence, $\Gamma_Q = \mathfrak{c}$, because the natural epimorphism $\pi_{PQ} : c_0/Q \to c_0/P$ induces a group epimorphism $\pi'_{PQ} : \Gamma_Q \to \Gamma_P$.

This consequence of $\mathsf{MA} + \neg\mathsf{CH}$ is also invoked in Proposition 6.25.

p. 124 l. 19. Replace "$a \in \mathbf{R}$" with "$a \in \mathbf{R} \setminus \{0\}$".

p. 152 l. -9. Replace "*ordinals*" with "*sets of ordinals*".

p. 152 l. -8. Replace "$\sigma_\alpha \leq \alpha$" with "$\sigma_\alpha \subseteq \alpha$".

p. 198 l. -4. Replace "$\aleph_2$" with "$\aleph_2 \cdot \mathfrak{c}$"—again, because of the mistake in Proposition 1.28(i).

p. 248. Replace the black arrowheads between the classes **ultrapower** and **cut prime z-ideal η_1**, and between the classes **ultrapower** and **hyper-real**, with gray arrowheads, thus changing "included and not equal" into "included and consistently not equal". As explained above, the proof on **p. 105** only has been shown to work under GCH.

p. 338. Replace the black arrowheads between the classes **ultrapower** and **cut prime z-ideal η_1**, and between the classes **ultrapower** and **hyper-real**, with gray arrowheads, thus changing "included and not equal" into "included and consistently not equal", see the remarks above on **pp. 105 and 248**.

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