Math 116c - Homework 2

Due: April 22, 2008 at 2:30 pm.

This Homework is due during lecture by Tuesday April 22 at 2:30 pm. Refer to the grading policy for additional requirements.

1. Show that ordinal exponentiation can be "intrinsically" defined as follows: $\alpha^{\cdot\beta}$ is the order type of the set

$$\{f \in {}^{\beta}\alpha : |\{\xi < \beta : f(\xi) \neq 0\}| < \omega\}$$

ordered lexicographically, i.e., for $f \neq g$ "finitely supported" functions from β to α, $f < g$ iff $f(\xi) < g(\xi)$ where $\xi < \beta$ is *largest* such that $f(\xi) \neq g(\xi)$ (in particular, check that this ordinal ξ exists).

2. (a) Prove Cantor's normal form theorem stating that any non-zero ordinal α can be represented in a unique way in the form

$$\alpha = \omega^{\cdot\beta_{n-1}} \cdot m_{n-1} + \cdots + \omega^{\cdot\beta_0} \cdot m_0$$

where $1 \leq n < \omega$, $\alpha \geq \beta_{n-1} > \beta_{n-2} > \cdots > \beta_0$, and $1 \leq m_i < \omega$ for all $i < n$.

(b) Show that $\alpha \geq \beta_{n-1}$ cannot be replaced with $\alpha > \beta_{n-1}$ by showing that there is a proper class of *epsilon numbers*, ordinals α such that $\alpha = \omega^{\cdot\alpha}$. Verify that the first such α is countable. It is usually denoted ϵ_0.

(c) Show that any ordinal $\alpha < \epsilon_0$ can be written in a unique way as $\alpha = \omega^{\cdot\beta} \cdot (\gamma + 1)$ where $\beta < \alpha$.

Definition 1. Define by transfinite recursion for each limit $\alpha < \epsilon_0$ an increasing ω-sequence $d(\alpha, n)$ cofinal in α by setting

$$d(\alpha, n) = \omega^{\beta}\gamma + \begin{cases} \omega^{\delta} n & \text{if } \beta = \delta + 1, \\ \omega^{d(\beta,n)} & \text{if } \beta \text{ is limit.} \end{cases}$$

(d) Verify that the sequences $(d(\alpha, n) : n < \omega)$ are well defined, strictly increasing, and that indeed $\sup_n d(\alpha, n) = \alpha$, for all limit ordinals $al < \epsilon_0$.

Definition 2 (Löb-Wainer)**.** The fast growing hierarchy

$$(f_\alpha : \alpha < \epsilon_0)$$

of functions $f : \omega \to \omega$ is defined by transfinite recursion as follows:

i. $f_0(n) = n + 1$.

ii. For $\alpha < \epsilon_0$, $f_{\alpha+1}(n) = f_\alpha^n(n)$, where the superscript indicates that f_α is iterated n times.

iii. For limit $\alpha < \epsilon_0$, $f_\alpha(n) = f_{d(\alpha,n)}(n)$.

(e) Show that $f_1(n) = 2n$, $f_2(n) = n2^n$, and $f_3(n)$ is larger than a stack of powers of two of length n.

(f) For $\alpha \leq \omega$ show that f_α is strictly increasing and that $\alpha < \beta \leq \omega$ implies that $f_\alpha(n) < f_\beta(n)$ for all but finitely many n.

As part of next week's homework set, you will show that item (f) holds for all $\alpha < \beta < \epsilon_0$. I will provide some hints, but I suggest that you start working on your own on how to prove this.