Math 116c - Homework 3

`Due:` April 29, 2008 at 2:30 pm.

This Homework is due during lecture by Tuesday April 29 at 2:30 pm. Refer to the grading policy for additional requirements.

1. The goal of this exercise is to prove the *Milner-Rado paradox*: For all cardinals $\kappa \geq \omega$ and all $\alpha < \kappa^+$ there are sets X_n, $n < \omega$, such that $\alpha = \bigcup_n X_n$ and $\mathrm{ot}(X_n) < \kappa^n$.

   (a) Show that if $0 < \rho < \kappa^+$ then $|\kappa^{\cdot\rho}| = \kappa$.

   (b) Conclude that it suffices to show the result for $\alpha = \kappa^{\cdot\rho}$, $\rho < \kappa^+$. This we will do by transfinite induction on ρ.

   (c) Show that if $0 < \rho < \kappa^+$, then we can write $\kappa^{\cdot\rho} = \bigcup_{\nu<\sigma} A_\nu$, where all of the following hold:

   - $\sigma = \kappa$ or $\mathrm{cf}(\rho)$.
   - For $\nu < \sigma$ there is $\rho_\nu < \rho$ such that $\mathrm{ot}(A_\nu) = \kappa^{\cdot\rho_\nu}$.
   - For $\nu < \tau < \sigma$, $A_\nu < A_\tau$, meaning that $\alpha < \beta$ for all $\alpha \in A_\nu$ and $\beta \in A_\tau$—equivalently, $\sup A_\nu \leq \min A_\tau$, with equality only if $\sup A_\nu \notin A_\nu$.

   (d) Use the induction hypothesis to check that we can write

   $$A_\nu = \bigcup_{n<\omega} A_{\nu,n}$$

   with $\mathrm{ot}(A_{\nu,n}) < \kappa^{\cdot n}$. Conclude the result when $\sigma = \omega$.

   (e) Conclude the result when $\sigma > \omega$ by considering $B_0 = \emptyset$ and $B_{n+1} = \bigcup_{\nu<\sigma} A_{\nu,n}$ for $n < \omega$.

The following two exercises use notation and results as in Homework 2. It is convenient to extend the definition of the sequence $(d(\alpha, n) : n < \omega)$ to successor ordinals $\alpha < \epsilon_0$ by setting $d(\beta + 1, n) = \beta$ for all $n < \omega$.

Definition 1. For $\alpha < \beta < \epsilon_0$, set $\beta \underset{n}{\to} \alpha$ iff there is a sequence $\gamma_0, \dots, \gamma_r$ such that $\gamma_0 = \beta$, $\gamma_r = \alpha$ and for all $i < r$, $\gamma_{i+1} = d(\gamma_i, n)$. Notice that $\gamma_0 > \cdots > \gamma_r$.

2. In this exercise we analyze some basic properties of the relation $\underset{n}{\to}$.

   (a) Let $\alpha, \beta < \epsilon_0$. Suppose that there are θ, δ such that $0 < \alpha = \omega^{\cdot\theta} \cdot \delta$ and $0 < \beta < \omega^{\cdot\theta+1}$. Show that $d(\alpha+\beta, n) = \alpha + d(\beta, n)$ for all $n < \omega$.

(b) i. Suppose that $\alpha \underset{n}{\to} \beta$, $\alpha \underset{n}{\to} \gamma$ and $\beta > \gamma$. Show that $\beta \underset{n}{\to} \gamma$.

ii. Suppose that $\alpha \underset{n}{\to} \beta \underset{n}{\to} \gamma$. Show that $\alpha \underset{n}{\to} \gamma$.

(c) Suppose that $\alpha \underset{n}{\to} \beta$ and $\lambda < \epsilon_0$ is a limit ordinal. Show that $\lambda + \alpha \underset{n}{\to} \lambda + \beta$.

(d) Let $0 < \alpha < \epsilon_0$ and $n < \omega$. Show that $\alpha \underset{n}{\to} 0$.

(e) Suppose that $k < l < \omega$, $\alpha < \epsilon_0$ and $n < \omega$. Show that $\omega^{\cdot\alpha} \cdot l \underset{n}{\to} \omega^{\cdot\alpha} \cdot k$.

(f) Let $n < \omega$ and $\delta < \epsilon_0$. Show that $\omega^{\cdot\delta+1} \underset{n}{\to} \omega^{\cdot\delta}$.

(g) Let $\alpha < \epsilon_0$ and $n < \omega$. Suppose that $\alpha \underset{n}{\to} \beta$. Show that $\omega^{\cdot\alpha} \underset{n}{\to} \omega^{\cdot\beta}$.

(h) Let $\lambda < \epsilon_0$ be a limit ordinal and let $i < j < \omega$. Show that $d(\lambda, j) \underset{0}{\to} d(\lambda, i)$.

(i) Let $\beta < \alpha < \epsilon_0$ and $i < n < \omega$. Suppose that $\alpha \underset{i}{\to} \beta$. Show that $\alpha \underset{n}{\to} \beta$.

(j) Suppose that $\epsilon_0 > \alpha > \beta$. Show that there is n such that $\alpha \underset{n}{\to} \beta$.

(k) Suppose that $\alpha \underset{n}{\to} \beta$ and $\alpha > \beta + 1$. Show that $\alpha \underset{n+1}{\to} \beta + 1$.

3. Now we apply the results of the previous exercise to show that each f_α ($\alpha < \epsilon_0$) is strictly increasing, and that if $\alpha < \beta < \epsilon_0$, then there is n such that $f_\alpha(m) < f_\beta(m)$ for all $m > n$. The relations $\underset{n}{\to}$ and the argument I indicate are due to Ketonen and Solovay. It is perhaps best to prove *simultaneously* several of the following statements (by induction):

(a) Let $\alpha < \epsilon_0$ and $n > 0$. Show that $f_\alpha(n) > n$.

(b) For $n > m$ and $\alpha < \epsilon_0$ show that $f_\alpha(n) > f_\alpha(m)$.

(c) Assume that $\alpha = \beta + 1 < \epsilon_0$. Show that $f_\alpha(n) \geq f_\beta(n)$ for all $n \geq 1$, and that the inequality is strict for $n > 1$.

(d) Show that if $\alpha \underset{n}{\to} \beta$ then $f_\alpha(n) \geq f_\beta(n)$.

(e) Suppose that $\epsilon_0 > \alpha > \beta$ and that $\alpha \underset{n}{\to} \beta$. Show that $f_\alpha(m) > f_\beta(m)$ for all $m > n + 1$. Conclude the desired result.