# The Knaster-Tarski theorem

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## 1 The result

Recall:

**Definition 1.1.** A *complete lattice* is a poset $\mathbb{P} = (P, \leq)$ such that any subset of $P$ has an inf and a sup.

The first part of the following theorem was shown in lecture. We include it to make this note self-contained.

**Theorem 1.2** (Knaster, Tarski)**.** *If $(\mathcal{L}, \leq)$ is a complete lattice and $f : \mathcal{L} \to \mathcal{L}$ is order-preserving (also called* monotone *or* isotone*), then $f$ has a fixed point. In fact, the set of fixed points of $f$ is a complete lattice.*

**Proof.** Let $M = \{x \in \mathcal{L} : x \leq f(x)\}$. Since $\mathcal{L}$ has a minimum, $M$ is nonempty. Since $f$ is order-preserving, $f''M \subseteq M$. Let $x \in M$ and $x_1 = \sup M$. Then $x \leq x_1$, so $f(x) \leq f(x_1)$ but also $x \leq f(x)$, so $f(x_1)$ is an upper bound for $M$. Hence, $x_1 \leq f(x_1)$, so $x_1 \in M$. But then $f(x_1) \in M$, so $f(x_1) \leq x_1$, and it follows that $f(x_1) = x_1$.

Let $S$ be the set of fixed points of $\mathcal{L}$. Clearly, $x_1 = \max S$. The same argument as above shows that if $x_0 = \inf\{x \in \mathcal{L} : f(x) \leq x\}$ then $x_0 = \min S$.

Let $\emptyset \neq X \subseteq S$. Let $y = \sup X$. Then $f(y)$ is also an upper bound for $X$, so $y \leq f(y)$. Now consider the set $\mathcal{S} = \{x \in \mathcal{L} : y \leq x\}$. With the order inherited from $\mathcal{L}$, $\mathcal{S}$ is itself a complete lattice. Let $f' = f \restriction \mathcal{S}$. Then $f'$ is an order-preserving map of $\mathcal{S}$ to itself. It follows that there is a *least* fixed point $y_1$ of $f'$. Clearly, $y_1 \in S$ and in fact $y_1 = \sup^S X$, where $\sup^S$ means that the sup is computed from the point of view of $S$ (but $y < y_1$ is possible). Similarly one shows that $y_0 = \inf^S X$ exists and therefore $(S, \leq)$ is a complete lattice—although, in general, it is not a complete sublattice of $\mathcal{L}$. □

Notice that to show that $S \neq \emptyset$ we only used that $\mathcal{L}$ has a minimum and every subset of $\mathcal{L}$ has a sup. Let us call such partially ordered sets *complete join semilattices*. The reason why we stated the existence of fixed points for complete lattices as opposed to complete join semilattices is the following observation:

**Lemma 1.3.** *Any complete join semilattice is a complete lattice.*

**Proof.** Let $(\mathcal{L}, \leq)$ be a complete join semilattice and set $0 = \min \mathcal{L}$. For $a \in \mathcal{L}$ let $[0, a] = \{x \in \mathcal{L} : x \leq a\}$. Let $\emptyset \neq A \subseteq \mathcal{L}$. Then $b = \sup \bigcap_{a \in A} [0, a]$ exists, and it is straightforward to verify that $b = \inf A$. $\square$

A *lattice* is a nonempty poset (with a max and a min) such that any finite subset has a max and a min.

**Theorem 1.4** (Davis)**.** *If $(\mathcal{L}, \leq)$ is a lattice such that any order-preserving function $f : \mathcal{L} \to \mathcal{L}$ has a fixed point, then $\mathcal{L}$ is a complete lattice.*

**Proof.** Assume that $\mathcal{L}$ is not complete. The argument proceeds by building a *gap*: Two sequences in $\mathcal{L}$, $(a_\xi : \xi < \alpha)$ and $(b_\nu : \nu < \beta)$, such that

- $a_\xi < b_\nu$ for all $\xi < \alpha$, $\nu < \beta$.
- $(a_\xi)_{\xi < \alpha}$ is strictly increasing.
- $(b_\nu)_{\nu < \beta}$ is strictly decreasing.
- There is no $c \in \mathcal{L}$ such that $a_\xi < c < b_\nu$ for all $\xi < \alpha$, $\nu < \beta$.

This is easily accomplished by an argument involving Zorn's lemma. We first claim that there exists a decreasing sequence $C \subseteq \mathcal{L}$ that has no inf. We show this by arguing that, otherwise, every subset $S \subseteq \mathcal{L}$ has a sup, so $\mathcal{L}$ is a complete join semilattice, contradicting Lemma 1.3: The proof is similar to the proof of Lemma 1.3. Let $U$ be the set of upper bounds of $S$, so $U \neq \emptyset$ since $\max \mathcal{L} \in S$. Let $C \subseteq U$ be a maximal decreasing sequence. Then $c = \inf C$ exists, by our assumption, and it is easy to see that $c = \sup S$.

We have shown that we can find a strictly decreasing sequence $(b_\nu)_{\nu < \beta}$ without an inf. Now consider a maximal increasing sequence $(a_\xi)_{\xi < \alpha}$ of lower bounds of $\{b_\nu : \nu < \beta\}$. Then $(a_\xi)_{\xi < \alpha}$ and $(b_\nu)_{\nu < \beta}$ are as required.

Now we define a function $f : \mathcal{L} \to \mathcal{L}$ and proceed to verify that $f$ is order-preserving and has no fixed points. Let $x \in \mathcal{L}$. Assume first that $x$ is a lower bound of $(b_\nu)_{\nu < \beta}$. Then there must be a least $\xi < \alpha$ such that $a_\xi \not\leq x$. Let $f(x) = a_\xi$ in this case.

If $x$ is not a lower bound of $(b_\nu)_{\nu < \beta}$, let $f(x) = b_\nu$, where $\nu < \beta$ is least such that $x \not\leq b_\nu$.

By definition, either $x \not\leq f(x)$ or $f(x) \not\leq x$, so $f$ has no fixed points. We conclude by showing that $f$ is order-preserving. For this, suppose that $x \leq y$. If $y$ is a lower bound of $(b_\nu)_{\nu < \beta}$, let $f(y) = a_\xi$. Then $a_\xi \not\leq x$, or else $a_\xi \leq y$. Hence, $f(x) = a_{\xi'}$ for some $\xi' \leq \xi$ (since $x$ is also a lower bound of $(b_\nu)_{\nu < \beta}$), so $f(x) \leq f(y)$.

We can thus assume that $y$ is not a lower bound of $(b_\nu)_{\nu < \beta}$. Let $f(y) = b_{\nu_0}$. If $x$ is not a lower bound either, then (since $x \leq y$), $f(x) = b_{\nu_1}$ for some $\nu_1 \geq \nu_0$, so $f(x) \leq f(y)$.

Finally, if $x$ is a lower bound, then $f(x) = a_\xi$ for some $\xi$, but then $f(x) < f(y)$. $\square$

## 2 Historical remarks

Theorem 1.2 was shown by Knaster and Tarski in 1927 for a particular case, see [3], namely, for any $S$, any $\subseteq$-increasing $\pi : \mathcal{P}(S) \to \mathcal{P}(S)$ has a fixed point. The proof of the Schröder-Bernstein theorem discussed in lecture (see Theorem 3.1) and applications to topology are mentioned in [3]. The general version [6] appeared in 1955 but Tarski mentions it in lectures from 1939. Around the time [6] was being prepared for publication, Tarski asked about the converse, and soon after Anne Davis proved Theorem 1.4. Davis's paper [1] appears in the same journal as Tarski's, immediately following his paper.

I followed [5] in the presentation of Lemma 1.3, but the result appears in the 1951 paper [2] of Jónsson and Tarski.

## 3 Applications

**Theorem 3.1** (Schröder, Bernstein)**.** *If there are injections $f : A \to B$ and $g : B \to A$ then there is a bijection $h : A \to B$.*

**Proof.** Consider $\pi : \mathcal{P}(A) \to \mathcal{P}(A)$ given by $\pi(X) = A \setminus g''(B \setminus f''X)$. Clearly, $\pi$ is $\subseteq$-preserving. Let $X_0$ be a fixed point of $\pi$. Then define $h = (f{\restriction}X_0) \cup (g^{-1}{\restriction}A \setminus X_0)$. □

The "usual" proof of Theorem 3.1 proceeds by explicitly building a fixed point $X_0$ by a back-and-forth argument, and then defining $h$ as above. For example, Kunen [4] sets $A_0 = A$, $B_0 = B$, $A_{n+1} = g''B_n$, $B_{n+1} = f''A_n$, and $A_\omega = \bigcap_n A_n$, and considers $X_0 = A_\omega \cup \bigcup_n (A_{2n} \setminus A_{2n+1})$. This corresponds to the *largest* fixed point of $\pi$. Other textbooks construct the smallest fixed point (by removing $A_\omega$ from $X_0$).

Of course, particular versions of Theorem 1.2 can be proved by building fixed points of the relevant maps by assuming some form of continuity and then iterating the maps. For example, one has:

**Theorem 3.2** (Tarski, Kantorovitch)**.** *Let $\mathbb{P} = (P, \leq)$ be a poset and let $f : P \to P$ be* sequentially continuous*, i.e., if $(c_n)_{n<\omega}$ has a supremum $c$, then $f(c) = \sup_n f(c_n)$. Assume that there is $b \in P$ such that $b \leq f(b)$ and any countable chain in $\{x : x \geq b\}$ has a supremum. Let $b' = \sup_n f^n(b)$, where $f^n$ is the $n^{th}$ iterate of $f$. Then $b'$ is a fixed point of $f$, and it is the least fixed point of $f$ in $\{x : x \geq b\}$.*

**Proof.** The obvious argument works. Just notice that $f$ is order preserving since if $y \geq x$ then $y = \sup\{x, y\}$, so $f(y) = \sup\{f(x), f(y)\}$. □

Other applications of Theorem 1.2 come from computer science (in denotational semantics), but here, again, one tends to be interested in "explicit" constructions of fixed points. Transfinite recursion allows us to do this: Consider an order-preserving $f : \mathcal{L} \to \mathcal{L}$ for $\mathcal{L}$ a complete lattice. Let $f^\alpha$ be the $\alpha^{\text{th}}$ iterate of $f$, so $f^0$ is the identity, $f^{\alpha+1} = f(f^\alpha)$ and for $a \in \mathcal{L}$, $f^\lambda(a)$, for $\lambda$

limit, is the sup of $\{f^\alpha(a) : \alpha < \lambda\}$. For any $a$, the sequence $(f^\alpha(a) : a \in \text{ORD})$ is (not necessarily strictly) increasing, so it eventually stabilizes, and necessarily this stable value is a fixed point of $f$. For $a = 0$ this gives the least fixed point of $f$. If instead we define $f^\lambda$, for $\lambda$ limit, as the infimum, then for $a = 1$ the sequence provides us with the largest fixed point.

An elaboration of the proof of Theorem 3.1 shows the following:

**Theorem 3.3** (Tarski)**.** *Let $\mathbb{B} = (B, \leq)$ be a complete Boolean algebra. Let $a, b \in B$ and let $f : [0, a] \to B$ and $g : [0, b] \to B$ be increasing. Then there are $a', b' \in A$ such that $f(a - a') = b'$ and $g(b - b') = a'$.* $\square$

This result has nice applications in the theory of complete Boolean algebras. We mention one from [6]: Say that two elements $a$ and $b$ of a Boolean algebra $\mathbb{B}$ are *homogeneous*, $a \approx b$, iff $([0, a], \leq)$ and $([0, b], \leq)$ are isomorphic.

**Theorem 3.4** (Tarski)**.** *Let $\mathbb{B} = (B, \leq)$ be a complete Boolean algebra. Let $a, b, c, a', c' \in B$ be such that $a \leq b \leq c$, $a' \leq c'$, $a \approx a'$ and $c \approx c'$. Then there is $b'$ such that $a' \leq b' \leq c'$ and $b \approx b'$.* $\square$

**Definition 3.5.** A *derivative algebra* is a structure $\mathcal{A} = (A, \leq, d)$ where $(A, \leq)$ is a Boolean algebra and $d : A \to A$ satisfies the following equations:

- $d0 = 0$.
- $ddx \leq x + dx$ for all $x \in A$.
- $d(x + y) = (dx) + (dy)$ for all $x, y \in A$.

(Where, of course, $x + y = \sup\{x, y\}$ and $0 = \inf A$.)

There is a standard example of derivative algebras, and a characterization theorem:

- Let $X$ be a topological space. Then $\mathcal{P}(X)$ is a derivative algebra with $\subseteq$ as $\leq$ and $dX$ the standard derivative of the set $X$, i.e., the set of all limit points of $X$.
- A *modal logic* is obtained by adding to propositional logic a *modal operator* $\square$ (read *it is necessary that*), so if $\varphi$ is a formula, so is $\square\varphi$. The modal logic $wK_4$ is defined by the axioms of propositional logic and the two modal axioms $\square(p \to q) \to (\square p \to \square q)$ and $(p \wedge \square p) \to \square\square p$, with inference rule Modus Ponens and $p \vdash \square p$. In the same sense that Boolean algebras correspond to propositional logic, derivative algebras correspond to the modal logic $wK4$. (This will be made a bit more precise later in the course, when we study Heyting algebras, which correspond to intuitionistic logic.)

Let $\mathcal{A} = (A, \leq, d)$ be a derivative algebra. An element $a \in A$ is *closed* iff $da \leq a$; it is *dense in itself* iff $da \geq a$; it is *perfect* iff $da = a$; and it is *scattered* iff there is no dense in itself element $x$ such that $0 \neq x \leq a$.

The following generalizes the Cantor-Bendixon theorem:

**Theorem 3.6** (Tarski)**.** *Let* $\mathcal{A} = (A, \leq, d)$ *be a derivative algebra such that* $(A, \leq)$ *is complete. Then every closed* $a \in A$ *admits a decomposition* $a = b + c$*,* $b \times c = 0$*, with* $b$ *perfect and* $c$ *scattered.*

**Proof.** Let $b = \sup\{x : a \times dx \geq x\}$ and let $c = a - b$. Then $a = b + c$ and $b \times c = 0$. Now let $d_a : A \to A$ be given by $d_a(x) = a \times dx$. Clearly, $d_a$ is order-preserving, so (by the proof of Theorem 1.2) $b$ is the largest fixed point of $d_a$: $b = d_a b = a \times db$. Then $b \leq a$ and $db \leq da$. But $a$ is closed, so $db \leq a$ and since $b = a \times db$, $b = db$, i.e., $b$ is perfect.

Let $x \leq c$ be dense in itself, so $dx \geq x$. Then $d_a x \geq x$ (since $a \geq c$), so $x \leq b$. Hence, $x = 0$. It follows that $c$ is scattered and we are done. □

# References


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[4] K. Kunen. **Set theory. An introduction to independence proofs**, Elsevier (1980).

[5] J. Nation. **Notes on Lattice Theory**, preprint. Available at `http://bigcheese.math.sc.edu/~mcnulty/alglatvar/lat0.pdf`

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