Continued fractions

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This term, Summer Hansen is taking a reading course with me on Pell’s equation. Our basic reference is:


We are allowing a rather organic approach to the material, taking as many detours as it seems appropriate. The topic of continued fractions naturally came up. What follows is just a short compendium of results I found interesting while reading on the subject.

1 Simple continued fractions

Throughout this note, $x$ is real and irrational.

Recall that the (simple) continued fraction of $x$ is given by

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}$$

where the $a_i$ are defined recursively using the following algorithm:

1. Let $x_0 = x$.
2. Given $x_n$, set $a_n = [x_n]$.
3. Let $x_{n+1} = \frac{1}{x_n - a_n}$.

We denote the continued fraction above by $[a_0; a_1, \ldots]$ and note that $a_i > 0$ for $i > 0$. The fraction is understood as the limit of the convergents $[a_0, \ldots, a_n]$, typically denoted $p_n/q_n$ with $q_n > 0$, $p_n, q_n \in \mathbb{Z}$ and relatively prime.

It is well known that the sequence $(p_{2n}/q_{2n} \mid n \geq 0)$ is strictly increasing and converges to $x$, while the sequence $(p_{2n+1}/q_{2n+1} \mid n \geq 0)$ is strictly decreasing, also converging to $x$.

If $x$ satisfies a polynomial equation of degree 2 with rational coefficients, we say that $x$ is a quadratic irrational. The following is a famous theorem of Lagrange:
**Theorem 1 (Lagrange)** The continued fraction of $x$ is eventually periodic iff $x$ is a quadratic irrational.

That $x$ is eventually periodic means that, in the notation above, there is a smallest positive integer $s$ such that $a_{n+s} = a_n$ for all $n$ sufficiently large, say $n \geq k+1$. We then write

$$x = [a_0; a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_{k+s}}]$$

and call $s$ the length of the period of $x$.


**Theorem 2** Suppose that $x = \sqrt{n}$ for some (non-square) positive integer $n$. Then we can take $k = 0$ in the expression above, i.e., $x = [a_0; a_1, \ldots, a_s]$. Moreover, $a_s = 2 \lfloor \sqrt{n} \rfloor = 2a_0$, and $a_1, \ldots, a_{s-1}$ is symmetric.

In fact, the theorem holds iff $n$ is a (non-square) positive rational. For example,

$$\sqrt{5} = [1; 3, 2]$$

and

$$\sqrt{1000} = [31; 1, 1, 1, 1, 6, 2, 2, 15, 2, 2, 15, 2, 2, 6, 1, 1, 1, 1, 1, 62].$$

In what follows, I restrict my attention to $x$ of the form $\sqrt{n}$, $n$ an integer.

It is natural to ask whether something can be said about the length of the period $s$, or about its parity.

### 2 The length of the period

About the first question, Sierpiński remarks that $s < 2n$. Significant improvements have been made here. For an example, I quote from MathReviews:

MR1042614 (91c:11045).
Rockett, Andrew M.; Szüsz, Peter.
*On the lengths of the periods of the continued fractions of square-roots of integers.*

On the basis of computer calculations [B. D. Beach and H. C. Williams, in Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, LA, 1971), 133–146, Louisiana State Univ., Baton Rouge, 1971; MR0321880 (48 #245); D. R. Hickerson, Pacific J. Math. 2...
it was once conjectured that $s$, the length of the period of the simple continued fraction expansion for $\sqrt{n}$, is $O(\sqrt{n})$.

The last upper estimate for $s$ is $O(\sqrt{n} \log(n/a^2))$, where $n/a^2$ is squarefree; it was found by R. G. Stanton, C. Sudler, Jr. and Williams [ibid. 67 (1976), no. 2, 525–536; MR0429724 (55 #2735)]. Using the unproven zeta-hypothesis, E. V. Podsypanin [Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 82 (1979), 95–99; MR0537024 (80h:12002)] has shown that $s = O(\sqrt{n} \log \log n)$.

In the present paper the authors prove that for “most” $n$, the relation $s = O(\sqrt{n})$ holds. The proof is based on Chebyshev’s inequality in probability theory and on Selberg’s upper bound sieve.

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Actually, Podsypanin’s result is somewhat more precise, but understanding his bound requires knowledge of algebraic number theory. Namely, he proves that if $s$ the length of the period of a quadratic irrational generating an order with fundamental unit $\varepsilon$, then

$$s < \log \varepsilon/(\log(1 + 5^{1/2})/2).$$

The bound mentioned above comes from estimating $\log \varepsilon$ (assuming the Riemann hypothesis).

Orders and units are discussed in detail in Chapter 4 of the Jacobson-Williams book. Here I just review the definitions.

Let $K$ be the field $\mathbb{Q}(\sqrt{n})$. There is a unique positive integer $a$ such that $n_0 = n/a^2$ is a square free integer. Let

$$\omega = \begin{cases} \frac{\sqrt{n_0}}{2} & \text{if } n_0 \not\equiv 1 \pmod{4}, \\ 1 + \frac{\sqrt{n_0}}{2} & \text{if } n_0 \equiv 1 \pmod{4}. \end{cases}$$

Clearly, $K = \mathbb{Q}(\omega)$. Recall that an algebraic integer is a number that is a root of a polynomial with integer coefficients. One can check that the algebraic integers that belong to $K$ are precisely the numbers of the form $a + b\omega$, where $a, b \in \mathbb{Z}$. The following is not the usual definition but coincides with it for our purposes:

**Definition 3** An order of $K$ is a subring $\mathcal{O} \subseteq K$ containing 1 and such that for some $m \in \mathbb{Z}$, $\mathcal{O} = \{a + bm\omega \mid a, b \in \mathbb{Z}\}$.

A unit of an order $\mathcal{O}$ is an element $\eta$ of $\mathcal{O}$ that divides 1, i.e., such that for some $\alpha \in \mathcal{O}$, we have $1 = \eta \alpha$. One can check that there is a unit $\varepsilon$ that generates all units in $\mathcal{O}$ in the sense that for any unit $\eta$ there is an $n \in \mathbb{Z}$ such that $\eta = \pm \varepsilon^n$. We call $\varepsilon$ the fundamental unit of $\mathcal{O}$.

In fact, let

$$\bar{\omega} = \begin{cases} -\frac{\sqrt{n_0}}{2} & \text{if } n_0 \not\equiv 1 \pmod{4}, \\ \frac{1 - \sqrt{n_0}}{2} & \text{if } n_0 \equiv 1 \pmod{4}, \end{cases}$$
be the conjugate of $\omega$. The discriminant of $O$ is $\Delta = m^2(\omega - \bar{\omega})^2 \in \mathbb{Z}$. Every element of $O$ has the form

$$x + \left(\frac{\Delta + \sqrt{\Delta}}{2}\right)y$$

for some integers $x, y$.

If $\eta$ is a unit, then $(2x + y\Delta)^2 - y^2\Delta = \pm 4$. Letting $X = 2x + y\Delta$, $Y = y$, and $\sigma = \pm 1$, accordingly, we see that

$$X^2 - \Delta Y^2 = 4\sigma.$$

From the basic theory of the Pell equation, there is a fundamental solution $(X_1, Y_1, \sigma_1)$ to this equation, in the sense that for any solution $(Z', Y', \sigma')$, we have

$$\frac{X' + Y'\sqrt{\Delta}}{2} = \pm \left(\frac{X_1 + Y_1\sqrt{\Delta}}{2}\right)^n$$

for some $n \in \mathbb{Z}$. We can take $\varepsilon = (X_1 + Y_1\sqrt{\Delta})/2$.

\section{The parity of $s$}

Results on the parity of the length $s = s(n)$ of the period of the continued fraction of $\sqrt{n}$ are very interesting. In $n = p$ is a prime number, it is a classical result that $s$ is odd iff $p = 2$ or $p \equiv 1 \pmod{4}$. (See also Aicardi's third paper below.)

When $n$ is not a prime, we do not have a full characterization of the parity of $s(n)$, but there is a significant amount of recent research in the area, starting with work of V.I. Arnold, who proved:

\textbf{Theorem 4} If $s(n)$ is odd, then $n$ is a sum of two squares. The converse does not hold.


Arnold’s results and conjectures have been extended by Francesca Aicardi in a series of papers:


The first two papers establish the theoretical framework used in the other two to obtain results on the parity of $s$. The terminology “red” is Arnold’s, who calls an integer red iff its period is odd.

Among the results obtained by Aicardi, I mention the following:

**Theorem 5**

1. If $n$ is divisible by a prime $p \equiv 3 \pmod{4}$, or by 4, then $s(n)$ is even.

2. If $s(n)$ is odd, then the sum of the elements of the period of the continued fraction of $\sqrt{n}$ is even. A partial converse holds: If the sum is even, and $n$ is prime, then $s(n)$ is odd.

Additional results require more notation. Given an irrational $x = a + b\sqrt{D}$ with $a, b \in \mathbb{Q}$, we can write

$$x = \frac{-k + \sqrt{k^2 - 4mn}}{2m}$$

for some integers $m, k, n$ relatively prime. The discriminant of $x$ is $\Delta = k^2 - 4mn$. For such $x$, write $s(x)$ for the parity of the length of the period of its continued fraction.

**Theorem 6** If $r, t \in \mathbb{Q}(\sqrt{n})$ are irrationals with the same discriminant, then $s(r)$ and $s(t)$ have the same parity.

**Theorem 7** Let $x \in \mathbb{Q}(\sqrt{n})$ be irrational, where $n$ is square free. Let $\Delta(x)$ be the discriminant of $x$. Then:

- If $n \equiv 3 \pmod{4}$, $s(x)$ is even,
- if $n \equiv 2 \pmod{4}$, then $s(x)$ is odd iff $\Delta(x) = 4l^2n$ for some integer $l$ such that $s(l^2n)$ is odd.
- If $n \equiv 1 \pmod{4}$, then $s(x)$ is odd iff $\Delta(x) = 4l^2n$ or $l^2n$ for some odd integer $l$ such that $s(l^2n)$ is odd.

I do not know of additional results, but numerical evidence suggests that the density of integers $n$ with $s(n)$ even is rather small. Aicardi computed $s(n)$ for $n < 50000$, finding that for about 3% of the integers $n$ in this range, we have that $s(n)$ is even.

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