# The 17 Plane Symmetry Groups

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## INTRODUCTION

Our paper is going to be about the 17 plane Symmetry Groups, also called the wallpaper groups, or the crystallographic groups. Each of the symmetry groups is made up of translations, rotations, reflections, and glide reflections, or any combination in between. We are going to talk about where the 17 Plane Symmetry Groups came from, how they came about, who discovered them, some prominent figures who used them in their work, what each Symmetry Group contains, and prove a couple theorems about the groups.

## HISTORY

The 17 plane symmetry group was discovered during the 19th century. The groups came from the discovery of the Pythagorean that there are five regular solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. These groups were used for different purposes. They were used heavily used by Egyptian craftsmen and in the Muslim world, as well as being used in architecture and decorative art.

Patterns have always been an interest to mankind, but in 1831 Hessel classified the 32 three dimensional point groups that correspond to the three dimensional crystal classes. They were later specified to the symmetrical groups of points which can have crystallographic symmetry by Frankenheim in 1835. It was not until 1891 when Evgraf Fedorov alongside German mathematician Schonflies and English geologist William Barlow proved that there were only 17 possible patterns in "The Symmetry of Regular Systems of Figures." Then in 1924, George Polya again proved that it was only possible to have 17 groups that did not have repetition within each other.

One prominent figure that used these 17 symmetry groups in his work was the artist M.C. Escher. Escher learned about tessellations and used the mathematical patterns and classifications in his work, and he specifically used hyperbolic tessellations that became apparent in his work. Escher became familiar with the symmetry groups due to a professor, the same person who proved the 17 groups the second time, George Polya. Polya's journal article struck Escher because it had pictures of each group in which Escher could see the different visual evidence of the ways that tiles or congruent shapes could fit together to satisfy one of the particular groups. However, not all of the 17 groups were new to Escher because many had been used in Islamic mosaics. The journal article proved so important to Escher that he copied all of Polya's illustrations into his copybook and identified it with the word, "Polya." The first success Escher had after coming across Polya's illustrations was a reptile motif.

## DEFINITIONS AND TERMINOLOGY

Now, in order to begin explaining all of the different groups we need to understand the four different types of motions that we can apply to our images. The first of these motions is a translation. In a translation everything in the image is moved by the same amount in the same direction. In other words, a translation shifts the image across the plane while preserving the position of all points. Translations are the only type of motion that is present in all of the seventeen groups we will be discussing. The next type of motion is a rotation. In a rotation one point of the image is fixed and the rest of the image shifts around that point. As the name implies, in a rotation, the image rotates about one fixed point. The third type of motion that can be applied to an image is a reflection. A reflection flips the image over a mirror line. The original image and the reflected image are an equal distance away from the mirror line. If we reflect an image twice with parallel mirror lines, this will result in a translation. However, if the mirror lines are not parallel, the result will be a rotation. These consequences will be proved later in this paper. The final type of motion that we will be discussing is a glide reflection. A glide reflection is a combination of a translation and a reflection. In a glide reflection, the first step is a mirror reflection then the image is translated parallel to the mirror line. Even though two steps are involved in obtaining the final product, this motion is considered to be one movement.

In order to understand the crystallographic groups, we must define some terminology. These definitions were found in an article by Doris Schattschneider which can be found in the reference section of this paper. A periodic pattern in the plane is a pattern where there is a finite region and two linearly independent translations such that the set of all images of the region when acted on by the group generated by these translations produces the original design and also there is a translation vector of minimum length that maps the pattern onto itself.

The next term that should be defined is translation group. A translation group of a periodic pattern is the set of all translations which map the pattern onto itself. In a periodic pattern there is a lattice of points. When any point in the pattern is chosen, the lattice of points is the set of all images of that point when acted on by the translation group of the pattern. The primitive cell is a unit which is a parallelogram whose vertices are lattice points. The primitive cell is very important when it comes to classifying groups and will be discussed more in the next section of this paper.

Lastly we will discuss the definition of a symmetry group. The symmetry group of the pattern is the set of all isometries which map the pattern onto itself. An important idea comes out of this definition. When a translation is performed on an image, its center of rotation is mapped to a new center of rotation which has the same order, therefore only rotations of 2, 3, 4, or 6, can occur as isometries of a periodic design. The proof of this can be seen later in this paper.

## NAMING OF GROUPS

The 17 plane symmetry groups are referred to by specific names. There are two predominant ways of naming the 17 plane symmetry groups. These two methods of naming the groups are the Orbifold Notation and the Crystallographic Notation. For the purposes of this paper we will be focusing mainly on the Crystallographic Notation. In this notation each group has a full length name and a shortened name. In the full length name there are 4 digits where each digit represents something about the group. Eleven of the seventeen groups have a shortened name. The shortened names are given whenever it is possible to shorten the name without causing confusion as to which group is being referred to.

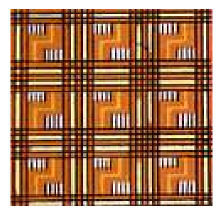
The first digit in the full length name depicts whether the image consists of primitive cells, which were defined earlier, or centered cells, which are chosen so that reflection axes will be normal to one or both sides of the cell as was described in the article by Schattschneider. Only two of the seventeen groups contain centered cells, the rest are primitive cells.

The next digit of the naming sequence denotes the highest order of rotation. What this means is that if the digit in the second place is a 1, the group contains no rotations because it is always in the identity position. If there is a 2 in the second place then that means that the group contains 180 degree rotations because it takes two rotations to get back to the identity and so on. It is important to remember that the second digit represents the highest order of rotation, and not the only order of rotation. For example, it is possible for the second digit of a group to be a 4 but the image may also have rotations of order 2. Examples of this can be seen later when each group is described in detail.

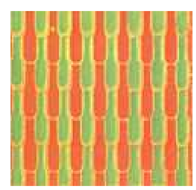
The third and fourth digits of each name refer to the symmetry axes of the group. In the third position there will be an m, g, or a 1. This digit refers to the symmetry axis normal to the x-axis. An m represents a reflection axis, a g represents no reflection but a glide reflection axis, and a 1 represents no symmetry axis. In the fourth position, the digit represents the symmetry axis at a specific angle to the x-axis. The angle here is dependent on n, which is our highest order of rotation. If n is a 1 or 2 then the angle is 180 degrees, if n is a 4 then the angle is 45 degrees, and if n is 3 or 6 then the angle is 60 degrees. In the fourth digit, there will once again be an m, g, or 1. These will represent the same thing that they did in the third digit. If there are no symbols in the third or fourth position it means that the group has no reflections or glide reflections.

**THE 17 PLANE SYMMETRY GROUPS**

The first of the seventeen groups is called p1, and it is the most simple of all the groups. By looking at its name, we can see that it has primitive cells and that its highest order of rotation is 1. This group contains only translations, and there are no reflections, rotations, or glide reflections in p1. The lattice of p1 is a parallelogram.



The full name of the second group is p211, but is usually shortened to p2. This group is very similar to p1 except that it contains 180 degree rotations. This is where the 2 comes from in its name because it takes two rotations of 180 degrees each in order to get the image back to the identity. This group also does not contain any reflections or glide reflections and the lattice type is also a parallelogram.



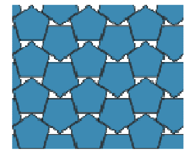
The third group is p1m1 or pm for short, and by its name we can tell that it has primitive cells, no rotations, but it contains mirror reflections. This is the first group that contains reflections but it does not contain any glide reflections. This is the first group that we have seen that does not have a parallelogram for its lattice. The lattice of this group is rectangular.

The fourth group is p1g1 or pg for short. Similar to the previous groups, this group has primitive cells but this group differs from the other groups because it is the first group that we encounter that contains a glide reflection with parallel axes. Since the second digit of the name contains a 1, we can see that there are no rotations present in this group. There are also no reflections in pg. The lattice of this group is rectangular like pm.

C1m1 is the fifth of the seventeen symmetric groups, and it is often referred to as just cm. This group is the first group that does not contain primitive cells, but this group has centered cells instead. This group contains no rotations, but it contains reflections and glide reflections. The reflection axes in this group are parallel. The rhombic lattice type of this group is different than any of the lattice types we have seen up to this point.

The sixth group is called p2mm or pmm. This group has primitive cells and contains 180 degree rotations whose fixed points lie on the axes of reflection. The reflection axes of this group are perpendicular. There are no glide reflections in this group and the lattice type is rectangular.

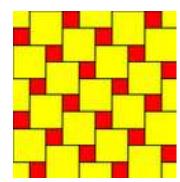
The next group is the seventh symmetry group and is called p2mg or pmg for short. This group contains primitive cells, 180 degree rotations (because rotating twice returns the image to the identity), reflections, and glide reflections. The reflections in this group are only in one direction and the axes of the glide reflections are perpendicular to the axes of the reflections. The centers of rotation all lie on glide reflection axes and the lattice type is rectangular.



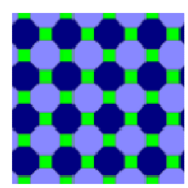
The next group is p2gg or just pgg for short. This group has primitive cells and also has 180 degree rotations. This group does not contain reflections but does have glide reflections present. The glide reflections occur in two perpendicular directions. The lattice type of this group is rectangular.

Besides cm, only one other group contains centered cells instead of primitive cells. This group is called c2mm or cmm for short. This group contains 180 degree rotations, reflections, and glide reflections. One characteristic of this group is that it has perpendicular reflection axes. The lattice type of this group is rhombic, as is cm. This group is frequently seen in everyday life because the most common technique of brick lying is done with this symmetry pattern.

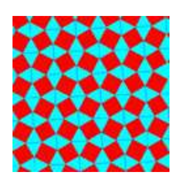
The tenth group is called p4. This is a group that has primitive cells and its highest order of rotation is 4, but it also contains rotations of order 2. This means that the image must be rotated four times in order to return to the identity therefore each rotation is 90 degrees. There are also 180 degree rotations present in this group. This is also the first group where we find a square as the lattice type. This group only contains translations and rotations, and there are no reflections or glide reflections present.



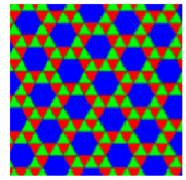
Next is p4mm or p4m. This group is similar to p4 in that it contains primitive cells and has 90 degree rotations, but it is different because it also contains reflections and glide reflections. In this group, all of the rotation centers lie on reflection axes. The lattice type in this group is square as well. Visually, this group looks like a grid.



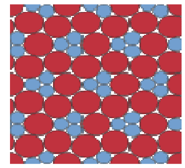
Our next group p4gm, or p4g, has many traits in common with the previous group, but in this group the rotation centers do not lay on the reflection axes and its axes of reflection are perpendicular. This group has a rotation order of 4 but also contains rotations of order 2. It also has a square lattice type, reflections, and glide



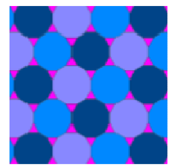
reflections.  The main difference between p4m and p4g is where the rotation centers lie.



The thirteenth group, p3, is the first group where we see a highest order of rotation of 3.  In this group, it takes three 120 degree rotations of the image to return to the identity.  This is also the first group that has a hexagonal lattice type. This group contains only translations and rotations, and there are no reflections or glide reflections present in this group.

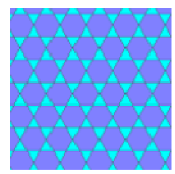


The group p31m, contains translations, rotations, reflections, and glide reflections.  The highest order of rotation is 3 and its lattice type is hexagonal. Both the reflection and glide reflection axes go in the three directions corresponding to an equilateral triangle.  A distinguishing characteristic of this group is that not all of the rotation centers must lie on a reflection axis.



The main difference between p31m and our next group, p3m1, is that in p3m1 all of the rotation centers must lie on the reflection axes.  This group contains primitive cells, has a highest rotation order of 3, contains both reflections and glide reflections, and has a hexagonal lattice.

The sixteenth group is called p6.  This is the first group where we see a highest rotation of order 6, but there are also rotations of order 2 and 3 present in this group.  This group does not contain reflections or glide reflections, and the lattice of this group is hexagonal.



Our final group is p6mm, or p6m for short.  This is the most complicated of all the groups.  Its highest order of rotation is 6, but it also contains rotations of order 2 and 3.  This group, however, does contain reflections and glide reflections. The lattice type of this group is hexagonal as well.

## DETAILED EXAMPLES

Here are some illustrated examples of wallpaper patterns that can be found in everyday life (Levine, 2008).

The pattern that appears above is a good example of a glide reflection isometry. It is essentially made up of one repeated figure that is aligned in two different ways. If the image is reflected across a horizontal line that cuts a row of the figures in half, then the result is that each figure is aligned in the opposite manner. The pattern can then be transposed so that it is identical to its original form. *Note: the same symmetry occurs across the vertical axis. This is an example of the group p4g as described above.*

This pattern is interesting because it has no reflection OR glide-reflection symmetry. It does, however, have rotational symmetry of order 6 at the center of hexagon-encased flowers, of order 3 at the centers of the grey triangles, and of order 2 at the midpoints between the hexagons. This is an example of the group p6 as described above.

## PROOFS

As was mentioned earlier, we are going to prove two different theorems about the 17 Plane Symmetry Groups. We found these proofs in two different articles. The full citations of references are included in the references section of this paper.

**Theorem 1**: *A reflection with mirror m followed by reflection with mirror m' is a translation when m is parallel to m' and a rotation otherwise.*

**Proof:** Let f and g be reflections with axes of reflection m and m', respectively. Also, let $\theta/2$ be the angle of reflection for f and let $\theta/2$ be the angle of reflection for g. So f = (v, $B_\theta$) and g = (w, $B_\varphi$). Then,

$$fg = (v, B_\theta)(w, B_\varphi) = (v + f_{B\theta}(w), B_\theta B_\varphi).$$

Suppose m and m' are parallel, then $\theta = \varphi$ and let u= v + $f_{B\theta}$(w). Then,

$$B_\theta B_\varphi = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

We find that, fg = (u, I), which is a translation.

Now, suppose m and m' are not parallel. Then,

$$B_\theta B_\varphi = \begin{bmatrix} \cos(\theta-\varphi) & \sin(\theta-\varphi) \\ \sin(\theta-\varphi) & -\cos(\theta-\varphi) \end{bmatrix} = A_{\theta-\varphi}$$

Therefore, fg = (u, $A_{\theta-\varphi}$), which is a rotation of $(\theta - \varphi)$ (Johnson & Rodriguez, 2003).

**Theorem 2**: *The only possible orders of rotation for a lattice of points are 2, 3, 4, and 6. (In other words, the only possible rotations are multiples of 60 or 90 degrees).*

**Proof:** It is easy to conceive of lattices with orders of rotations of 2, 3, 4, or 6. Thus, it is only necessary to prove that a rotation of order of 5 or of an order greater than 6 is impossible. We choose an arbitrary center of rotation A of order n. Since we have some sort of lattice, let B be another center of rotation on the lattice of minimal distance from A. Since all of the points on the lattice are identical with respect to rotational symmetry, we know that B has rotational symmetry of order n. Hence, we can find a point A' by a $2\pi/n$ rotation around B and another point B' by a $2\pi/n$ rotation around A'. Because these transformations are isometries, we know that

$$AB = A'B = A'B' = L.$$

The question that remains is the relation between AB, AB' and AA', which we will denote as X and Y respectively. But since B was chosen to be the minimal lattice distance from A, it follows that

$$AB \leq AB' \text{ and } AB \leq AA' \text{ or rather that } L \leq X \text{ and } L \leq Y.$$

If n = 5, then, A, B, A', and B' form a trapezoid ABA'B'. Since $2\pi/5 < 2\pi/4$, we have:

$$X = A'B\ (1 - 2\cos 2\pi/5) < A'B\ (1 - 2\cos 2\pi/4) = A'B = L.$$

Thus, we have L > X, a contradiction of minimality, and the contradiction proves that rotational symmetry of order 5 is impossible.

Now if n > 6, then we employ the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos 2\pi/n$. Since ABA' is an isosceles triangle, we rewrite it as

$$Y = \sqrt{2L^2\left(1 - \frac{\cos 2\pi}{n}\right)} < \sqrt{2L^2\left(1 - \frac{\cos \pi}{3}\right)} = L.$$

Again we have a contradiction, and it is clear that it is impossible for a lattice to have rotational symmetry of order n > 6 (Levine, 2008).

## CONCLUSION

As shown and described above, the 17 Plane Symmetry Groups can be found in everyday life whether we notice the patterns or not. By researching this topic, we have expanded our knowledge about the history of the 17 groups, how each group is named, how each group is described, and a couple of different applications of proofs. This paper has helped us to identify each pattern, and we now catch ourselves searching out these patterns in their many different forms that surround us in the real world.

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