The Banach-Tarski Paradox

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1. Introduction

In 1924, Stefan Banach and Alfred Tarski Published an article in *Fundamenta Mathematicae*, entitles, “Sur la decomposition des ensembles de points en parties respectivement congruents” (Feferman and Solomon 43). The main result of this article asserted that given any three dimensional sphere,in theory, the sphere can be cut up into a finite number of pieces, and be reassembled to form a ball of any size. This means the original sphere could be cloned or a sphere the size of a pea can be assembled to be the size of the sun. However, this result breaks normal logic. Normal intuition tells us that a sphere can not be reassembled into anything larger than the original sphere. Due to this contradiction of basic logic, the theorem became known as the Banach-Tarski Paradox. This proposed idea was eventually proven to be consistent with the axioms of set theory and shown to be non-paradoxical.

1.1 Biographies in (Very) Brief

Stefan Banach was born on March 30th, 1892 in Krakow. His mother was unable to support him and he was sent to live with friends and family. Banach started attending school at the age of 10 and was always caught doing math problems during breaks with his friends. He ran into a man named Hugo Steinhaus who gave him the opportunity to work with him at a university. It was here that Banach did most of his research and where he was awarded his doctorate for a paper defining basic principles of functional analysis.

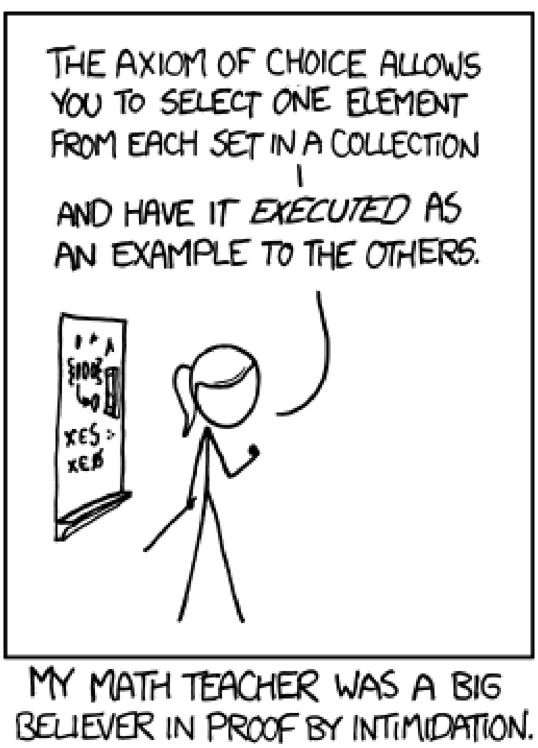
Alfred Tarski was also born in Poland and entered into the university as soon as he could. Even though he entered with the intent to study biology, he was soon spotted to have quite a bit of potential as a mathematician and was persuaded to change his degree. After becoming one of the youngest people to receive a doctorate from the university, he began traveling and meeting other mathematicians.

1.2 Banach-Tarski Paradox

The Banach-Tarski Paradox says that a solid three-dimensional ball can be decomposed into a finite number of pieces and rearranged in such a way that the original ball can be reassembled into two identical copies of the original. This result seems impossible. The key difference that makes this possible is that a mathematical object is what is being talked about. A mathematical ball has an infinite number of points (infinite density) whereas a physical ball has a finite number of points (density based on finite and countable number of molecules). If we assume radius one, our ball would be defined as $S = \{(x,y,z) |\ x^2 + y^2 + z^2 \leq 1\}$.

The simple explanation of the paradox is that intuitive notions of “size” break down when applied to infinite sets. This is true even for relatively simple sets such as the integers. For example, if the integers are split into two subsets, one containing all even integers and the other containing all odd integers, both subsets are still infinite. Furthermore, there exists a bijection from the original set Z to *each* subset, so they are *both* as “big” (that is, they have same cardinality) as the original integers. Intuition suggests that by splitting the integers into two sets, which are each as large as the original, we have doubled the size of the set! This logic begs a better definition of “size” when applied to sets, and requires in the case of Banach-Tarski a “splitting” process that is so complex as to be indescribable in concrete terms. Before attempting to make several three-dimensional balls from one using a splitting process, some foundational ideas and theorems should be discussed.

2. Axiom of Choice



Proof of Zermelo’s well-ordering theorem given the Axiom of Choice:

1: Take S to be any set.

2: When I reach step three, if S hasn’t managed to find a well-ordering relation for itself, I’ll feed it into this wood chipper.

3: Hey, look, S is well-ordered.

(from the webcomic series xkcd.com)

2.1 Statement of the Axiom of Choice

The Axiom of Choice was first formally stated by Ernst Zermelo in 1904, although it had been invoked by mathematicians previously without explicit treatment of its implications. Formally, it states that:

> for any family of nonempty sets $(S_i)_{i \in I}$ there exists a family of elements $(x_i)_{i \in I}$ such that $x_i \in S_i$ for each $i \in I$.

To aid understanding, one can use a physical analogy: say a warehouse is filled with crates, and none of the crates are empty. Then it is possible to select one object from each crate and thus create a collection of "representative" objects.

This analogy helps explain the meaning of the symbols above but is fairly simplistic, for any such warehouse in the physical world would only contain a finite number of crates, and the process of selecting a representative object from each crate - although tedious - would be straightforward. Thus the fact that there exists a selection process, or even several possible processes, is obvious in the real world. The Axiom of Choice is similarly trivial when applied to finite collections of sets. Indeed, in the extreme case where the family is composed only of a single set, the Axiom of Choice is merely an awkward restatement of the notion that every nonempty set has an element, an idea which is provable by other means.

2.2 Choice Functions and Choice Sets

It is useful to consider the "selection process" as a **choice function** f defined on a family of nonempty sets, such that for every nonempty set S_i in the family, $f(S_i) \in S_i$. In that case, the Axiom of Choice states that for every family of nonempty sets, such a choice function f exists. When applied to a single set, the choice function provides an element; when applied to families of sets, the choice function yields an ordered tuple - one out of all possible Cartesian products of the distinct sets in the family - which displays the "representative element" selected from each set in order. So alternatively, if every set in the family is nonempty, the Axiom of Choice states that the Cartesian product of the family must be nonempty. Sometimes instead of a tuple, the elements are organized into a **choice set** without ordering constraints. (Wikipedia, Axiom of Choice)

In addition to these formulations of the Axiom of Choice there are many others. Rather than simply continuing to list alternative statements, it is beneficial to view some examples of choice functions in order to understand when the Axiom is necessary to prove their existence.

Below are examples of choice functions which do not rely on the Axiom of Choice:

- Take the set of integers and partition it into subsets such that each subset contains exactly two consecutive elements. Propose a function which takes from the subsets

all even elements. Since every subset has exactly one even element, and every even element relates to a different subset, it is clear that such a function is a choice function. Better yet, it is known in advance exactly how the resulting tuple (or set) will look.
- Take the natural numbers and partition it arbitrarily into subsets such that none are empty. Propose a function which takes the smallest element from each subset. Because the natural numbers are well ordered, every subset has a smallest element by definition, and the function is a choice function.
- Take a finite set and partition it arbitrarily into subsets such that none are empty. Propose a function which takes a random element from each subset. Since there are at most a finite number of subsets and thus a finite number of “selection” steps, the process will eventually complete, and the resulting tuple can be confirmed to contain exactly one element from each set. Thus even without special selection rules, the finite number of sets dictates that a choice function is attainable. This is equivalent to the “crates in a warehouse” and is a choice function, with or without the Axiom of Choice.

By contrast, the real numbers are neither finite nor automatically well-ordered; that is, there exist subsets of the reals such as (0,1) in which there is no smallest number. Thus, if one partitions the real numbers arbitrarily into infinitely many subsets, there is usually no rule available which can be verified to be a choice function, and any set-by-set process of “selecting” would never end. Still, each set does have elements to offer to the tuple, so why couldn’t one exist? This notion that it is possible to imagine such a tuple when it is impossible to construct and verify one is the essence of the Axiom of Choice. The connection to well-ordering is no coincidence, as Zermelo formalized the Axiom of Choice to support his theorem that the reals can be well-ordered. Commonly called Zermelo’s Theorem or the Well-Ordering Theorem, it is logically equivalent to the Axiom of Choice. (Wikipedia, Well-Ordering Theorem)

2.3 Role in Axiomatic Set Theory

In 1931 Kurt Gödel showed that the existing axioms of Zermelo-Fraenkel (ZF) set theory cannot disprove the Axiom of Choice (Wikipedia, Kurt Gödel), and Paul Cohen later showed that the axioms of ZF set theory also cannot prove the Axiom of Choice (Wikipedia, Paul Cohen). Therefore, the Axiom of Choice is logically independent from ZF set theory, and mathematicians use the shorthand ZFC to denote Zermelo-Fraenkel set theory recognizing the Axiom of Choice, and ZF$\neg$C (or just ZF) to denote a Zermelo-Fraenkel set theory which omits the Axiom of Choice. Proofs which are viable only through the application of the Axiom of Choice are non-constructive, since although the Axiom asserts the existence of a choice function, it gives no detail about how one might be created nor can it explicitly describe the properties of such an object. Like proving existence by contradiction, the method is rejected by strict constructivists. (In fact,the “law of the excluded middle,” which allows proof of existence by contradiction, is derivable from the Axiom of Choice.) For more on this topic, see Constructivism, Law of Excluded Middle, and Axiom of Choice.

In addition to constructivist objections, the fact that the Axiom of Choice can lead to paradoxical results has historically generated controversy over its acceptance. The Axiom of Choice allows selection of non-measurable infinite sets, such as a Vitali set (Wikipedia, Vitali Set). Specifically, the Axiom allows selection of the non-measurable sets which underly the decompositions of the Banach-Tarski Paradox, but in the process renders it impossible to define which elements each set should contain, due to the non-constructive aspect of the Axiom. Although the majority of modern mathematicians accept the Axiom of Choice for its utility (or through recognition of some equivalent statement) it is still considered beneficial when a result obtained through Axiom of Choice can be proven in a framework other than ZFC.

The results either derived from the Axiom of Choice or logically equivalent to it span many branches of mathematics. A list of notable examples can be found on the Wikipedia.org article for Axiom of Choice, but a few concepts already familiar to undergraduates are listed below:

- Zermelo's Well-Ordering Theorem uses the Axiom of Choice to show that every set (notably, the real numbers) can be well-ordered. This provides the justification for transfinite induction.
- Every vector space has a basis.
- The Law of Trichotomy extends to the cardinality of sets.
- Every surjective function has a right inverse.
- Every union of countable sets is countable.

In each example, it is easy to see how the infinite nature of the result eludes a purely constructive approach. Although these concepts are not always taught to undergraduates with explicit credit to the Axiom of Choice, they are not provable (or at best take a much weaker form) in any ZF framework which excludes the Axiom of Choice (Wikipedia, Axiom of Choice)

3. Splitting the Surface: The Hausdorff Paradox

The Hausdorff paradox was named after Felix Hausdorff and states that if you remove a certain "countable" subsets of the sphere, S^2, the remainder can be divided into three disjoint subsets A, B, and C such that each is congruent to the others and all are congruent to the set $B \cup C$. In other words, there is a countable subset of D of S^3 such that S^3/D is SO_3 paradoxical (i.e. $SO^3$is the group of rotations of $R^3)$(Wagon, 18). In particular, it follows that on S^2there is no finitely additive measure (see **Lebesgue measure** below) defined on all subsets such that the measure of congruent sets is equal (because this would imply that the measure of A is both ⅓ and ½ of the non-zero measure of the whole sphere) (wikipedia, Hausdorff paradox). This paradox was first published in Mathematische Annalen in 1914 and also in Hausdorff's book, Grundzuge der Mengenlehre that same year (wikipedia, Hausdorff paradox).

This paradox shows that there is no finitely additive measure on a sphere defined on *all* subsets which is equal on congruent pieces. Hausdorff first showed the easier result that there is no *countably* additive measure defined on all subsets. The structure of the SO^3group plays a crucial role here, the statement is not true on the plane or the line. In fact, it was later shown by Banach that it is possible to define an "area" for all bounded subsets in the Euclidean plane(as well as "length" on the real line) such that congruent sets will have equal "area". This Banach measure is the "area" which can be defined for every set--even those without a true geometric area--which is rigid and finitely additive (Wolfram Mathworld, Banach Measure). So it is not a measure in the full sense, but it equals the Lebesgue measure which is the standard way of assigning a measure to subsets of n-dimensional Euclidean space. Lebesgue measure has the properties that are useful: If A is a disjoint union of countably many disjoint Lebesgue measurable sets, then A is itself Lebesgue measurable and λA is equal to the sum(or infinite series) of the measure of the involved measurable sets and Lebesgue measure is strictly positive on non-empty open sets and so its support is the whole of R^n(wikipedia, Lebesgue Measure).

This implies that if two open subsets of the plane (or real line) are equidecomposable then they have equal area. So any infinitely sized set when split into subsets will result in more infinitely sized subsets. This is a concept that is hard to grasp because mostly we live in a world of finite area based on finite distribution of matter, as opposed to sets such as the reals which are dense or sets like the integers which have no distinctly extreme elements. In other words, when we divide infinity by a finite number we still have infinity, an operation that defies intuition and elementary physics.

3.1 Banach-Schroder Theorem

The Banach-Schroder Theorem suggests the following: Suppose G acts on X and A,B $\subseteq X$. If A$\preceq$B and B$\preceq$A, the A $\sim_G B$. Thus $\preceq$is a partial ordering of the $\sim_G -classes$ in P(X). The proof for this is as follows. Proof: The relation $\sim_G$ is easily seen to satisfy the following two conditons:

(a) If A $\sim$ B then there is a bijection g:A$\rightarrow$B such that C$\sim$g(C) whenever C$\subseteq$A and,

(b) If $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, and if $A_1 \sim B_1$and $A_2 \sim B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$
.

The rest of the proof assumes only that $\sim$is an equivalence relation on P(X) satisfying (a) and (b).

Let f: A$\rightarrow B_1$, g: $A_1 \rightarrow B$where $B_1 \subseteq B$ and $A_1 \subseteq A$, be bijections as guaranteed by (a). Let $C_0 = A \setminus A_1$ and, by induction, define C_{n+1} to be $g^{-1}f(C_n)$; let C= $\bigcup_{n=0}^{\infty} C_n$. then it is easy to check that g(A\C) = B\f(C), and hence the choice of g implies that A\C~B\f(C). But, by choice of f, C~f(c) and property (b) yields (A\C) $\cup$C ~ (B\f(C)) $\cup$f(C), or A~B as desired. QED. This proof essentially supports the idea of equidecomposability. (Wagon)

A proposition (for the that is also closely related to this is idea states the following: Suppose G acts on X and E, E' are G-equidecomposable subsets of X. If E is G-paradoxical, so is E' (Wagon). We will call this proposition one in the interest of referencing it for our proof. This proposition is important but on its own is fundamentally flawed. On its own, this proof does not restrict subset size. Due to this, there is no guarantee that A and B have the same area. The Banach-Schroder Theorem addresses this concern.

The last major theorem related to the Banach-Tarski Paradox is a theorem (we will call it theorem 2). It states: If D is a countable subset of S^2, then S^2 and $S^2[\backslash \mathrm{backslashD?}]$ are S $O_3 - equidecomposable$ (using two pieces) (Wagon). The proof is as follows. Proof: We seek a rotation, p, of the sphere such that the sets D, p(D), $p^2(D)$,...are pairwise disjoint. This suffices, since then $S^2 = \overline{D} \cup (S^2 \setminus \overline{D})$~p($\overline{D}$) $\cup (S^2 \setminus \overline{D}) = S^2[\backslash \mathrm{backslashD?}], where\ \overline{D} = \bigcup \{p^n(D): n = 0,1,2,...\}$. Let l be a line through the origin that misses the countable set D. Let A be the set of angles θ such that for some n > 0 and some $P \epsilon D$, p(P) is also in D where p is the rotation about l through nθ radians. Then A is countable, so we may choose an angle θ not in A; let p be the corresponding rotation about l. Then $p^n(D) \cap D = \oslash$ if n > 0, from which it follows that whenever $0 \leq m < n$, then $p^m(D) \cap p^n(D) = \oslash$ (consider $p^{n-m}(D) \cap D)$); therefore p is required. QED (Wagon) This theorem leads right into the Banach-Tarski Paradox.

4. Banach-Tarski Revisited

The Banach-Tarski Paradox states that S^2 is $SO_3 - paradoxical$ as is any sphere centered at the origin. Moreover, any solid ball in $R^3 is\ G_3 - paradoxical$ and R^3 itself is paradoxical. The proof for this is as follows.

Proof: The Hausdorff Paradox states that S^2\S is $SO_3 - paradoxical$ for some countable set D (of fixed points of rotations). Combining this with the previous theorem and Proposition 3.4 yields that S^2 is $SO_3 - paradoxical$. Since none of the previous results depends on the size of the sphere, spheres of any radius admit paradoxical decompositions.

It suffices to consider balls centered at **O,** since G_3 contains all translations. For definiteness, we consider the unit ball *B*, but the same proof works for balls of any size. The decomposition of S^2 yields one for B\{**O**} if we use the radical correspondence: P $\rightarrow$ {$\alpha P: 0 < \alpha \leq 1\}$. Hence it suffices to show that B is $G_3 - equidecomposable$ with B\{**O**}, that is, that a point can be absorbed. Let P=(0,0,1/2) and let *p* be a rotation of infinite order about an axis through *P* but missing the origin. Then, as usual, the set D = $\{p^n$(**O**): n$\geq$0} may be used to absorb **O**: *p*(D) = D\{**O**}, so B $\sim$ B\{**O**}. If, instead, the radial correspondence of S^2 with all of R^3 \{**O**} is used, one gets a paradoxical decomposition of R^3\{**O**} using rotations. Since, exactly as for the ball R^3 \{O} $\sim_{G3}$ R^3, R^3 is paradoxical via isometries. QED. (Wagon)

This version of the proof relies on an infinite amount of available rotations. Rotations are a class of permutation which preserve volume, and this is why the theorem has come to be known as a paradox. There is a lot of controversy surrounding this theorem. The result is so counterintuitive that it seems that one of the fundamental assumptions must be false. The Axiom of Choice is usually labeled as that assumption.

4.1 Generalized Result

A stronger version of the Banach-Tarski Paradox states that if A and B are any two bounded subsets of R^3, each having nonempty interior, then A and B are equidecomposable. The proof is as follows. Proof: It suffices to show that $A \preceq B$, for then the same argument, $B \preceq A$ and Theorem 3.5 yields $A \sim B$. Choose solid balls K and L such that $A \subseteq K$ and $L \subseteq B$, and let *n* be large enough that L may be covered by *n* (overlapping) copies of L. Now, if S is a set of *n* disjoint copies of L, then using the Banach-Tarski Paradox to repeatedly duplicate L, and using translations to move the copies so obtained, yields that $L \succeq S$. Therefore $A \subseteq K \preceq S \preceq L \subseteq B, so\ A \preceq B$. QED. This result has been used to do some work within the fields of topology and to prove some results on finitely additive measures.

4.2 The von Neumann Paradox

There were some results from the Banach-Tarski Paradox that were very valuable to the world of mathematics and one of these phenomenons was studied by John von Neumann. John von Neumann gave a conceptual explanation between the planar and higher-dimensional cases. Unlike the group of SO_3 of rotations in three-dimensions, the group E_2 of Euclidean motions is solvable, which implies the existence of a finitely-additive measure on E_2 and R^2 which is invariant under translations and rotations, and rules out paradoxical decompositions non-negligible(Wikipedia, Banach-Tarski Paradox).

The von Neumann paradox is the idea that one can break a planar figure such as the unit square into sets of points and subject each set to an area-preserving affine transformation such that the result is two planar figures of the same size as the original. This was proved by von Neumann in 1929 assuming the axiom of choice, he realized the trick of such so-called paradoxical decompositions was the use of a group of transformations which include as a subgroup a free group with two generators. The group of area preserving transformations contains such subgroups, and this opens the possibility of performing paradoxical decompositions using them (Wikipedia, Von Neumann Paradox).

The class of groups isolated by von Neumann in the course of study of Banach-Tarski phenomenon turned out to be very important for many areas of mathematics: these are amenable groups with an invariant mean, and include all finite and all solvable groups.

Generally speaking, paradoxical decompositions arise when the group used for equivalences in the definition of equidecomposability is not amenable. Von Neumann's paper left open the possibility of a paradoxical decomposition of the interior of the unit square with respect to the linear group $SL_2 - paradoxical,$ or perhaps even F-paradoxical, where F is the free group generated by $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and it's transpose(Wagon, 101).

5. Summary for the Layman

Dispensing with the formal mathematics, one can describe the process of the proof of the Banach-Tarski Theorem in a few simple steps. (For thorough but accessible proof with illustrations, see Wapner, 143-156.)

Step 1: The Group of Rotations

There are two basic rotations, the clockwise rotation of 120 degrees and a clockwise rotation of 180 degrees, set about different axes. These two rotations can be combined sequentially in a countably infinite number of ways to yield infinitely many other rotations. All such rotations would be grouped into one set with infinite size (cardinality). We then begin partitioning this set into 3 subsets, by using the uniqueness theorem which states that every rotation in our original set has a unique, reduced form of representation (a "word" naming in order the rotations required, with no consecutive instances of a rotation and then its inverse). We see that the process of subdividing our original set would be infinite. If we denote our subsets as A, B, C we can see by the axiom of choice we want to subdivide our subgroups so that each and every rotation in B is related to a specific rotation in A and every rotation in B U C is related to a specific relation in A. Thus the cardinality of A, B, and B U C are all equivalent.

Step 2: Partitioning the unit Sphere into Two Copies

In this step we take the unit sphere, the surface of the unit ball, separate and reassemble it into two sets, each of which is identical in shape to the original unit sphere. Notice that every rotation in our original set has two poles (poles are the points that stay stationary when the sphere rotates). So if there is a countable infinity of different rotations, there exist countably many poles on the unit sphere. So there is a set of all poles associated with the original set of rotations on the sphere.

Then if the poles are removed, there is a set of all other "leftover" points on the sphere. Although both the set of poles and the set of "leftovers" are infinite, the cardinality of sets dictates that when the set of poles (countable) is removed from the sphere (uncountable) the "leftovers" are uncountable, and thus much more numerous (more common or probable, depending on your preference) than the set of poles. This is where the Axiom of Choice comes in. We need to choose a set containing all those points which are related to both our original set of rotations and the set of all points on the sphere. We know they coincide, but cannot effectively describe every such point in the set. This set is uncountably infinite, has no points in common with the set of poles, and no point in the set can be rotated to any other point in the set by one of the rotations in the original set of rotations.

So there are four disjoint infinite sets which include all the points on the surface of the sphere: three sets of “leftovers” and one set of the poles. All the points on the sphere can be separated into three disjoint sets by applying the three subsets of the original set of rotations from step 1. By construction, these four disjoint sets can be rotated by basic rotations to achieve congruent sets (that is, every “word” can be concetenated with some other “word” to create cancellations) so that if we denote these sets again as above in step 1 we have $A \simeq B \simeq C \simeq B \cup C$. Since A U B U C is the entire uncountable surface of the sphere, A contains roughly 1/3 of the surface; but simultaneously, A is congruent to (B U C) and both B and C are congruent to A, so now any given set among the three contain simultaneously $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$ of the sphere surface! The Hausdorff Paradox comes into play here since it allows us to separate the sphere into a finite number of pieces which can be reassembled to form two spheres, each identical in shape to the original. This technique uses B U C as a cutting template which can be placed directly onto each A, partitioning each into two sets, one congruent to set B and the other being congruent to set C. So we now have six congruent subsets which represent all the points on the original sphere except for the set of poles. Here we use the set of poles from the original sphere to plug the holes representing the missing poles in one of the two copies of the sphere; for the other copy we use the concept of equidecomposability to conclude that the surface of any sphere which is missing only a countable number of points is equidecomposable to the complete sphere.

Step 3: Extending the Process to the Unit Ball

We can now extend our sphere into a ball by taking the original sphere and imagining an inward thickening, extending from the surface of the sphere smaller “skins” up to but not including the center of the sphere.

Recall that in step 2 we established the Hausdorff Paradox and the creation of the two spheres. Intuitively we can see that all the congruencies from step 2 will still hold for the thickened sets. The extension to a “punctured ball” missing the center point is clear, then: we can separate and rearrange the punctured ball into two copies, each piece-wise congruent to the original. Now we must deal with the center of the two punctured congruent balls to create two solid balls from one. As we did with poles, we can use the original center of the ball to complete one of the two copies, and shift from infinity to fill the second copy.

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