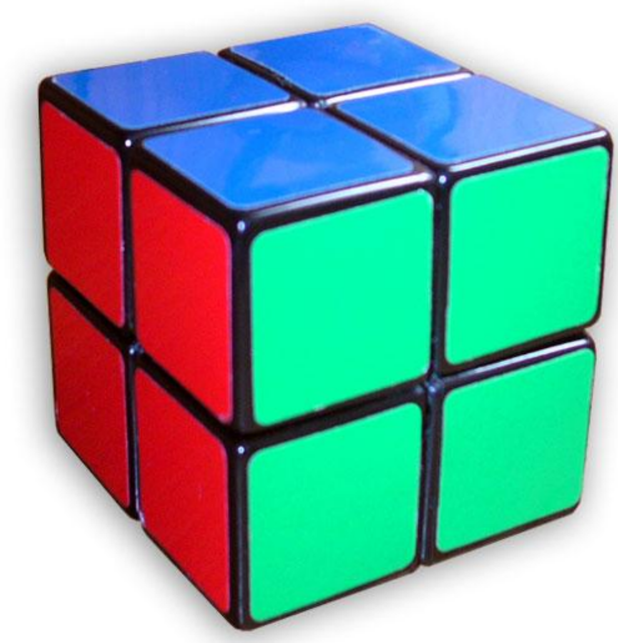
The Pocket Cube



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Introduction

When most people hear about the Rubik's Cube, they think about the 3 x 3 x 3 game cube that was invented in 1972 by Hungarian architect, designer, and university professor named Ernő Rubik. What many people do not know is the man who is credited with inventing the cube was not the first person to come up with the idea. In March 1970, two years before the patented Rubik's cube came about, a Canadian man by the name of Larry Nichols invented a 2 x 2 x 2 "Puzzle with Pieces Rotatable in Groups," which would soon become the Pocket Cube. The fist cube was held together with magnets, and did not become all the rage until a few years later. Eventually Ernő Rubik beat Nichols in the race to get the 2 x 2 x 2 cube patented on March 29, 1983. There are many people across the globe that compete to achieve a world record time in solving the Pocket Cube. According the World Cube Association, a person by the name of Christian Kaserer is the current world record holder in solving the Pocket Cube in competition, with a time of 0.69 seconds set at the Trentin Open 2011. The person with the best average time of five solves is Feliks Zemdegs, with a world record time of 2.12 seconds set at the Melbourne Cube day 2010. Solving the cube goes far beyond luck and speed, there are many mathematical theories that can help a person solve and understand the cube.

Numbers to Consider

The Pocket cube is composed of 8 small cubes, with 3 orientations each, giving a maximum of $8! \cdot 3^8$ positions. Then there are the 3 total twists of the cube, and the orientation of the puzzle does not matter. So there are $7! \cdot 3^6 = 3{,}674{,}160$ possible positions of the cubes. The 2 x 2 x 2 cube has six sides, or faces, each face has four facets, for a total of $6 \cdot 4 = 24$ facets.

Notation of the Cube

When working with the cube, it is essential to determine a way of referring to the faces of the cube. The most logical notation was developed by British mathematician David Singmaster. Once an orientation of the cube is fixed in space, the six faces are then labeled right (R), left (L), up (U), down (D), front (F), and back (B). It is said that the advantage of naming the faces this way is so that each face can be referred to by a single letter. The faces are labeled in the following way:

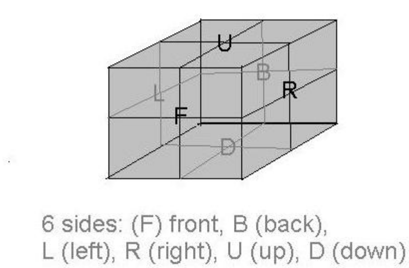


Figure 1: Faces of a 2 x 2 x 2 cube

(Source: http://www.permutationpuzzles.org/rubik/webnotes/rubik.pdf)

Written in clockwise order, the Singmaster notation is:

Front face	fru	frd	fld	flu
Back face	blu	bld	brd	bru
Right face	rbu	rbd	rfd	rfu
Left face	lfu	lfd	lbd	lbu
Up face	urb	urf	ulf	ulb
Down face	drf	drb	dlb	dlf

The 2 x 2 x 2 Rubik's Cube Forms a Group

We have learned in class about what it takes for a set with a binary operation to be considered a group. There are four axioms in which the set must obey:

1. Closure- For every $x_1, x_2 \in G$, then $x_1 \cdot x_2 \in G$.
2. Associativity- For every $x_1, x_2, x_3 \in G$ we have $x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$.
3. Identity- There must exist an element $id \in G$ such that for every $x \in G$, we have $id \cdot x = x \cdot id = x$.
4. Inverse- For every $x \in G$, there is an element $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = id$.

So the question remains, does the Rubik's cube form a group? First we need to define the operation as being a move that rotates one face of the cube by 90° clockwise. The elements of group are sequences of moves, which relate to permutations of the faces of the individual cubes. In order for the Rubik's cube to be considered a group, we need to make sure all four the axioms listed above are satisfied:

- ✓ Closure- No matter what move is carried out, we still have a cube.
- ✓ Associativity- $F(RL) = (FR)L$.
- ✓ Idenitity- The identity is making no moves on the cube, or $F^4 = id$.
- ✓ Inverse- By doing the moves backwards we get back to the identity. For example, $(FRBL)(L^{-1}B^{-1}R^{-1}F^{-1}) = id$. Essentially, the operation is equivalent to function composition, which is associative.

From this we can say that the Rubik's cube is a group.

Permutation Puzzle

From the book, *Adventures in Group Theory*, author David Joyner discusses the procession of permutation puzzles. He first begins by explaining what a one-person game is. Joyner describes a one-person game as "a sequence of moves that follow certain rules":

- There are finitely many moves at each stage.
- There is a finite sequence of moves that yields a solution.
- There are no chance or random moves.
- There is complete information about each move.

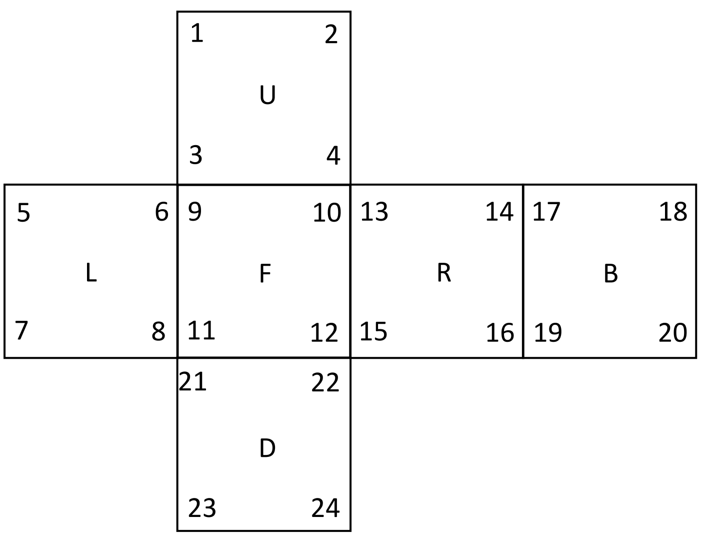
- Each move depends only on the present position, not on the existence or non-existence of a certain previous move. (Joyner, 61)

Joyner then moves on to "nailing down" the definition of a permutation puzzle. He depicts a permutation puzzle as "a one-person game with a finite set T of puzzle pieces" that satisfies the following four properties:

(1) For some $n > 1$ depending on the construction of the puzzle, each move of the puzzle corresponds to a unique permutation of the numbers in $\mathbb{Z}_n$.
(2) If the permutation of $\mathbb{Z}_n$ in (1) corresponds to more than one puzzle move, then the two positions reached by those moves have to be distinguishable.
(3) Each move, for example F, must be 'invertible,' such that there is another move, F^{-1}, which brings the puzzle back to the original position it was in before the move F was done.
(4) If F is a move that corresponds to the permutation g_1 of T and if B is a move that corresponds to the permutation g_2 of T, then $F * B$ is either
    - ⇨ Not a legal move, or
    - ⇨ Corresponds to the permutation $g_1 * g_2$. (Joyner, 62)

From this we can see that the 2 x 2 x 2 cube satisfies the properties of a permutation puzzle. The moves fixing the orientations of the corners of the cube correspond to $\mathbb{Z}_3$. Also, each move is "invertible," that is, if you do the move F, one only has to do F^{-1} to get the cube back into its original position.

We will label the 24 facets of the 2 x 2 x 2 Rubik's Cube as follows:



From this we can give the standard generators, corresponding to the six faces of the cube, written in disjoint cycle notation:

$$U = (1,2,4,3)(9,5,17,13)(10,6,18,14)$$

$$L = (5,6,8,7)(9,21,20,1)(11,23,18,3)$$

$$R = (13,14,16,15)(10,2,19,22)(12,4,17,24)$$

$$B = (17,18,20,19)(14,1,7,24)(16,2,5,23)$$

$$D = (21,22,24,23)(11,15,19,7)(12,16,20,8)$$

$$F = (9,10,12,11)(3,13,22,8)(4,15,21,6)$$

Algorithm for Solving the Cube

First, one must decide which color should be placed on the top face. Next, hold the cube so that at least one piece on the top face shows that color. The other pieces on the top face will be placed relative to that piece. Find a piece in the down layer (D) that belongs in the top, or upper, layer (U). Now, rotate D so that the piece is below its destination and rotate the whole cube to get it to the front right face. If all the U layer pieces are already in the top layer, though some could be placed incorrectly, then choose any D piece to displace any wrong piece from the U layer. Now all that is left is fixing the orientation of the individual cubes. Depending on the orientation, do one of the following:

- To move the cube from the frd to the urf position do the move FDF^{-1}.
- To move the cube from the rdf to the urf position do the move $R^{-1}D^{-1}R$.
- To move the cube from the dfr to the urf position do the move $FD^{-1}F^{-1}R^{-1}DDR$.

To completely solve the top layer repeat the previous steps.

After solving the first layer, if, for example, the top layer is supposed to be yellow then twist the top layer until there is a yellow in the position 3 (when referring to a numbered position, refer back to the numbered drawing created above). Now that the yellow square is in position 3, use the following algorithm:

(1) $RUR^{-1}URUUR^{-1}$

This may need to be repeated, and for each time the move is made, make sure the yellow square is in position 3. After doing this move a few times the top layer will be all yellow but the corners may not be in the desired position. From this move, when done repeatedly, two of the top corners should be oriented correctly. Now, twist the top layer until the two properly oriented corners are in the 10 and 12 positions and the two incorrectly oriented corners should be in the 9 and 11 positions. Then complete the next algorithm,

(2) $RU^{-1}L^{-1}UR^{-1}U^{-1}L$

This will make the top layer look disfigured again, but just repeat the first algorithm until the cube is solved.

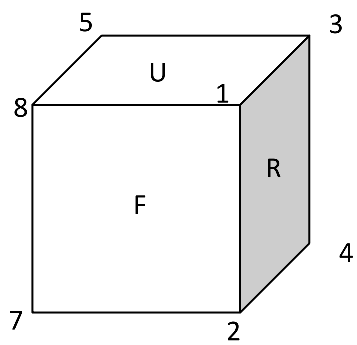
Listed below are the permutations that result after performing each algorithm from the solved state of the cube:

(1) $(13,18,4,5,10,1)(14,3,2,9,17,6)$
(2) $(13,18,2,9)(10,1,17,6)(14,3,4,5$

Again, these numbers are referring to the numbers that were assigned to each face of the cube.

The Second Fundamental Theorem of Cube Theory

We will label the vertices of the cube as follows:



Let X represent one of the valid moves:

$$F, R, U, B, L, D$$

Let $\rho(X) \in S_8$ be the permutation of the vertices.

Also, let $v(X)$ be the orientation of the corners, where each v_i is the number of clockwise twists required to return that vertex to its relative orientation.

Let H by the group generated by the moves F, R, U, B, L, D. If $g, h \in H$, then

(1) $v(gh) = v(g) + \rho(g)^{-1}(v(h))$

This is somewhat clear, if we perform the move g then the orientation of the corners would be $v(g)$. Now we want to imagine leaving the corners with their new orientation by going back to the orgianl permutation so we can see how the move h affects the orientation. So if $\rho(g)$ send the vertex $i \to j$, then we will do the move $\rho(g)^{-1}$ and compose that with the orientation of $v(h)$

This give us $v(gh) = v(g) + \rho(g)^{-1}v(h)$.

Now we want to show that for any list of moves $x_1 \ldots\ldots x_k$ the "conservation of total twists" will hold.

We will prove this by induction on the number of moves we make, where each move $x_i \in F, R, U, B, L, D$.

$i = 0$: None of the vertices change position or orientation.

$i = 1$:

X	$v(X)$
F	$(2,0,1,0,2,0,0,1)$
B	$(0,1,0,2,0,1,2,0)$
R	$(1,2,2,1,0,0,0,0)$
L	$(0,0,0,0,1,2,2,1)$
U	$(0,0,0,0,0,0,0,0)$
D	$(0,0,0,0,0,0,0,0)$

Induction Hypothesis:

Assume $i = k$ satisfies the conservation of total twists. So, by equation (1),

$v(x_1 \ldots\ldots x_k) = v(x_1 \ldots\ldots x_{k-1})\rho(x_1 \ldots\ldots x_{k-1})v(x_k)$.

Now, $i = k + 1$. So we have that

(2) $v(x_1 \ldots\ldots x_{k+1}) = v(x_1 \ldots\ldots x_k)\rho(x_1 \ldots\ldots x_k)v(x_{k+1})$

The term $v(x_1 \ldots\ldots x_k)$ satisfies the "conservation of total twists" by induction hypothesis.

The term $\rho(x_1 \ldots\ldots x_k)v(x_{k+1})$ satisfies the "conservation of total twists" because x_{k+1} is a valid move $F, R, U, B, L, or\ D$ which all satisfy "conservation of total twists." Also, $\rho(x_1 \ldots\ldots x_k)$ was defined as the permutation of the vertices and therefore does not change the orientation.

Since both terms in equation (2) are congruent to $0\ mod\ 3$,

$v(x_1 \ldots\ldots x_k)\rho(x_1 \ldots\ldots x_k)v(x_{k+1}) \equiv 0\ mod\ 3$. □

Orders of Elements of the 2 x 2 x 2 Rubik's Cube

In the article, "Group Theoretic Properties of the Rubik's Cube," author Evan Cordell discusses the orders of elements of the 2 x 2 x 2 Rubik's Cube. Using software that was written in C_{++} he came to several discoveries:

- Any individual operation has order 4.
- Any permutation, or move, of two bordering sides (such as RB) has order 15.
- Any permutation of two nonadjacent sides (such as UD) has order 4.
- Any permutation of three operations adjacent at the corner (such as FRU) has order of either 30 or 60.
- Any permutation of three nonadjacent sides (such as FLB) has order 24.

References

[1] D. Joyner, *Adventures in Group Theory: Rubik's Cube, Merilin's Machine, and Other Mathematical Toys*. The Johns Hopkins University Press, 2008.

[2] D. Singmaster, *Notes on Rubik's Magic Cube*. Enslow, 1981.

[3] E. Cordell, "Group Theoretic Properties of the Rubik's Cube.".

[4] H. Poole, "Group Theory and the Rubik's Cube.".

[5] W.D. Joyner, "Mathematics of the Rubik's Cube.".

[6] "Pocket Cube," Wikipedia. http://en.wikipedia.org/wiki/Pocket_Cube.