THE BANACH-TARSKI PARADOX AND AMENABILITY

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Abstract. The Banach-Tarski paradox famously states that (using only isometries) it is possible to "disassemble" a sphere into multiple pieces then "reassemble" those pieces into two spheres. This paper presents a proof of this result and relates it to the concept of an amenable group. Along the way, we will see that this result is very natural and intuitive, despite its surface appearance.

1. Introduction

The Banach-Tarski paradox states that solid spheres (in $\mathbb{R}^3$) permit a paradoxical decomposition; that is, it is possible to partition a solid sphere and apply isometries to the pieces of the partition to yield two solid spheres of same radius as the original. It is clear why this result is known as the Banach-Tarski *paradox*; how is it possible to duplicate a sphere using only measure-preserving functions? While this looks like a logical paradox, it is not, as is clear from reading and understanding the proof. Rather, the result follows from other, more intuitive, theorems in a very satisfying manner. The overall theorem may look contradictory, but each step is (relatively) easy to follow.

The approach I will take to prove the Banach-Tarski paradox is due to Wagon[1]. In addition to the general layout of the theorems, most proofs uses ideas from Wagon's book.

2. Group actions and paradoxical decompositions

Before we can prove the Banach-Tarski paradox, it is necessary to formally define the notions of paradoxical sets and sets being equidecomposable with each other. Of course, both of these rely on groups acting on a set.

Definition 1. A group G is said to be a *group action* on set X (equivalently, G acts on X) if for each $g \in G$ there is a bijection $g : X \to X$ such that

- For any $g, h \in G$ and $x \in X$, $g(h(x)) = (gh)(x)$, and
- For any $x \in X$, $1(x) = x$, $1 \in G$ being the identity.

We are now ready to define paradoxical decompositions and equidecomposability.

Definition 2. Let G act on X and $Y \subseteq X$. We say Y is *G-paradoxical* (or, if the group is clear, merely *paradoxical*) when for positive integers m, n there exist pairwise disjoint subsets $A_1, \ldots, A_m, B_1, \ldots, B_n$ partitioning Y with $g_1, \ldots, g_m, h_1, \ldots h_n \in G$ such that

$$Y = \bigcup_{i \leq m} g_i(A_i) = \bigcup_{j \leq n} h_j(B_j).$$

Note that without loss of generality, it may be assumed that $g_i(A_i)$ (respectively $h_j(B_j)$) are disjoint.

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Less formally, Y can be partitioned into two (finite) families of subsets. For each of these families, each set can be transformed by some element of G such that the (transformed) family covers Y. That is, G can be broken into pieces then reassembled into two copies of G.

Definition 3. Let G act on X with $A, B \subseteq X$. We say A and B are *G-equidecomposable*, written $A \sim_G B$ (or $A \sim B$, if the group is clear) if A and B can each be partitioned into $n \in \mathbb{N}$ pairwise distinct subsets $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ such that for $g_1, \ldots, g_n \in G$, for each $i \leq n$, $g_i(A_i) = B_i$.

Less formally, A can be broken apart and rearranged to yield B and vice versa. This also allows the following characterization of paradoxicality:

Fact 4. *Let G act on X. $Y \subseteq X$ is G-paradoxical if and only if there exist disjoint $A, B \subseteq Y$ such that $Y \sim_G A$ and $Y \sim_G B$.*

Proof. This is immediate from the definitions of paradoxicality and equidecomposability. Note that even though we do not require $A \cup B = Y$, we can take $Y \setminus (A \cup B)$ as a set in the paradoxical decomposition of Y with 1_G (the identity of G) acting on it. Obviously, this will not affect the set covered by the union of the group actions on the partition of A. □

The usage of the symbol $\sim$ to denote equidecomposability is justified as it is an equivalence relation.

Fact 5. *$\sim_G$ is an equivalence relation. Further, it satisfies the following two properties:*

- *If $A \sim B$ there is a bijection $b : A \to B$ such that if $A' \subseteq A$, then $A' \sim b(A')$, and*
- *If A_1, A_2 and B_1, B_2 are pairs of disjoint sets with $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$.*

Proof. Let G act on X with $A, B, C \subseteq X$. To see that $A \sim A$, merely notice that $A = 1_G(A)$. To see that $A \sim B$ implies $B \sim A$, let A_i, B_i, g_i witness $A \sim B$. Then, A_i, B_i, g_i^{-1} witness $B \sim A$.

For transitivity, let A_i, B_i, g_i $(i \leq m)$ witness $A \sim B$ and B_j, C_j, h_j $(j \leq n)$ witness $B \sim C$. Now, partition each A_i into n pieces based upon which set B_j contains $g_i(a)$ for $a \in A_i$. That is, $A_{i,j} = \{a \in A_i : g_i(a) \in B_j\}$. Note that some of these sets may be empty. In such a case, we can ignore them when showing $A \sim C$, so assume no $A_{i,j}$ is empty, without loss of generality. Similarly, partition each C_j into m pieces based upon which set B_i contains $h_j^{-1}(c)$ for $c \in C_j$. That is, $C_{j,i} = \{c \in C_j : h_j^{-1}(c) \in B_i$. Again, assume without loss of generality that each $C_{j,i}$ is nonempty. Now, for each i, j, $h_j(g_i(A_{i,j})) = C_{j,i}$ as $B_i \cap B_j = g_i(A_{i,j}) = h_j^{-1}(C_{j,i})$, by construction of $A_{i,j}, C_{j,i}$. That is, $A \sim C$, so $\sim$ is an equivalence relation.

We now show $\sim$ satisfies the properties above. To prove it satisfies the first property, assume that $A \sim B$ (with A_i, B_i, g_i witnessing) and let $A' \subseteq A$ be given. For each A_i, define $A'_i = A' \cap A_i$. Now, define $B'_i = g_i(A'_i)$. Obviously, the bijection we need must map each A'_i to B'_i, whence it maps A' to $B' = \bigcup B'_i$. However, it must do so for any $A' \subseteq A$ simultaneously. As we use the same g_i for any $A' \subseteq A$, $b(a) = g_i(a)$ (where $a \in A_i$) satisfies this condition. b is a bijection as each g_i is a bijection.

Proving $\sim$ satisfies the second property is even easier. Let $A_{1,i}, B_{1,i}, g_{1,i}$ witness $A_1 \sim B_1$ and $A_{2,j}, B_{2,j}, C_{2,j}$ witness $A_2 \sim B_2$. As $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, merely combine the partitions to get partitions of $A_1 \cup A_2$ and $B_1 \cup B_2$. The same elements of G as above witness $A_1 \cup A_2 \sim B_1 \cup B_2$. □

These two properties show that $\sim$ behaves reasonably with regard to set theoretic operations; in a rather loose sense, it is preserved under disjoint union and subsets. This allows a very natural partial order $\preceq$ on sets such that $A \sim B$ if and only if $A \preceq B$ and $A \succeq B$. The most well-known order with these properties is cardinality; e.g. $|A| \leq |B|$ if and only if there's a bijection from A to a subset of B. The famous Cantor-Schröder-Bernstein theorem states that this order is well-defined. However, it can be extended to other relations which satisfy the same two conditions. In general, though, this order is not total. Whereas every set has the same cardinality as a cardinal and the cardinals are well-ordered, there is not necessarily a "backbone" of sets ordered by equidecomposability.

Theorem 6 (Banach-Cantor-Schröder-Bernstein)**.** *Let $\sim$ be an equivalence relation on sets with the properties that*

- *If $A \sim B$ there is a bijection $b : A \to B$ such that if $A' \subseteq A$, then $A' \sim b(A')$, and*
- *If A_1, A_2 and B_1, B_2 are pairs of disjoint sets with $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \cup A_2 \sim B_1 \cup B_2$.*

Then, there is a natural partial order $\preceq$ such that $A \preceq B$ if $A \sim B' \subseteq B$. In particular, if $A \preceq B$ and $B \preceq A$, then $A \sim B$.

This will be essential in proving the strengthening of the Banach-Tarski paradox.

Proof. Standard in the literature. In the less-general case of this theorem with "has the same cardinality" as the equivalence relation, the only two proprieties of this relation used are the two above. Hence, removing any extraneous mention of cardinality transforms a proof of the Cantor-Schröder-Bernstein theorem to a proof of the more general case. For a proof specifically of the general case, see e.g. Chapter 3 of [1]. □

We are now ready to prove the main result.

3. The Banach-Tarski paradox

We will prove that solid spheres in $\mathbb{R}^3$ are paradoxical with respect to the group of isometries G_3 by steps. First we prove that if the free group of rank 2 (denoted by F) acts on a set without nontrivial fixed points, then that set is F-paradoxical. Then we show that $F \leq SO_3$, the group of rotations of $\mathbb{R}^3$ (which is obviously a subgroup of G_3). This allows us to show that all but a countable portion of the unit sphere S^2 is SO_3-paradoxical. This result is commonly known as Hausdorff's paradox and is crucial to proving Banach-Tarski. Using a trick, this countable gap can then be filled, which shows that a solid sphere without its center is G_3 paradoxical (if centered at the origin, it is SO_3-paradoxical). Finally, this single point is accounted for by the previous trick, finishing the proof. The strong form of the Banach-Tarski paradox follows an immediate corollary.

While it may be possible to prove this more directly by giving an explicit decomposition of the solid sphere, this approach better illuminates the underlying logic. As the goal is to understand why this result is true—not merely to be convinced that it is—this is the preferable approach. The indirectness of our approach does obscure some details (e.g. the exact number of pieces needed for the decomposition), but that is a small price to pay.

Before this devolves into amateur musings on the nature and purpose of mathematics, let us move on to the proving the necessary lemmas. Many of these are interesting and important results in their own right; in the interests of not writing a book, I will not delve into them deeper than is necessary for our purposes.

Theorem 7. *Let G acting on X be G-paradoxical and have no nontrivial fixed points in X. Then, X is G-paradoxical.*

Proof. Let $g_i, b_j \in G$, $A_i, B_j \subseteq G$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ witness that G is paradoxical. Let M be the choice set containing exactly one element of each orbit of G in X. Consider the set $M' = \{g(M) : g \in G\}$. We claim this partitions X. Obviously, $\bigcup M' = X$ as for $m \in M$, $G(m)$ is the orbit of m, and the orbits of a group action exhaust the set the group acts on. To show that sets in M' are pairwise disjoint, assume otherwise towards a contradiction. Then, there are $m, n \in M, g, h \in G$ $(m \neq n)$ such that $g(m) = h(n)$. Hence, $m = g^{-1}h(n)$, so $m = n$; if $m \neq n$, they are in different orbits and hence there is no action which maps n to m. Therefore, $g^{-1}h$ fixes m. This contradicts G's lack of nontrivial fixed points.

Now, for each i, j let $A'_i = \bigcup_{g \in A_i} g(M)$ and $B'_j = \bigcup_{h \in B_j} h(M)$. As M' is a partition of X and the A_i and B_j sets are pairwise disjoint, the A'_i and B'_j sets are also pairwise disjoint. Then, $X = \bigcup_i g_i(A'_i) = \bigcup_j h_j(B'_j)$, due to the bijection $g \mapsto g(M)$ from G to M' and the paradoxicality of G. □

This theorem is important solely because of the following special case, which will be crucial in proving Hausdorff's paradox:

Corollary 8. *If F is a free group of rank 2 acting on X with no nontrivial fixed points, then X is F-paradoxical.*

Proof. By Theorem 7, all that must be shown is that F is F-paradoxical. Let σ, τ be the free generators of F. Denote by $W(w) \subseteq F$ be the set of all words which begin with w on the left. Then, clearly, $F = \{1\} \cup W(\sigma) \cup W(\sigma^{-1}) \cup W(\tau) \cup W(\tau^{-1})$, where each of these sets are pairwise disjoint. Note that this simply says that every nontrivial word in F begins with $\sigma^{\pm}$ or $\tau^{\pm}$. Further, if $w \in F$ does not begin with σ (respectively τ), then $\sigma^{-1}w \in W(\sigma^{-1})$ (respectively for τ), so $w = \sigma\sigma^{-1}w \in \sigma W(\sigma^{-1})$. That is, $F = W(\sigma) \cup \sigma W(\sigma^{-1}) = W(\tau) \cup \tau W(\tau^{-1})$, which shows that F is F-paradoxical, completing the proof. □

Of course, if $F \leq G$ and X is F-paradoxical, then X is G-paradoxical; its decompositions uses elements of F which are elements of G. Hence, to show that all but a countable subset of S^2 is SO_3-paradoxical we would like that $F \leq SO_3$. Fortunately, this happens to be the case!

Theorem 9. *SO_3 (and hence also G_3) contains a free subgroup of rank 2.*

Proof. We show there exist two independent rotations about the axes through the origin of $\mathbb{R}^3$. Let σ, τ be rotations through the z-axis and x-axis respectively, both of angle $\arccos \frac{1}{3}$. Then, their matrix representations in the standard basis are

$$\sigma = \begin{pmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

To show that these rotations (and their inverses) generate a free group of rank 2, we must show that no nontrivial reduced word in $\sigma^{\pm 1}$ and $\tau^{\pm 1}$ is the identity. Consider such a word w. Due to the strong symmetry of $\mathbb{R}^3$, we may assume without loss of generality that the rightmost term of w is $\sigma^{\pm 1}$. For every word ending in $\sigma^{\pm 1}$, there is a corresponding word ending in $\tau^{\pm 1}$; merely swap every σ with τ and vice versa. For example, if $w = \tau^2\sigma$, the corresponding $w' = \sigma^2\tau$. That is, there is an automorphism a on this group with $a(\sigma) = \tau$ and $a(\tau) = \sigma$. Similarly, there is an automorphism a' on this group with $a'(\sigma) = \sigma^{-1}$, $a'(\tau) = \tau^{-1}$. Hence, we can further assume that w ends in σ.

Fix nontrivial reduced w with rightmost term σ is the identity. We claim that $w(1,0,0) = (a, b\sqrt{2}, c)/3^k$, for $a, b, c, k \in \mathbb{Z}$ with $3 \nmid b$. Hence, $w(1,0,0) \neq (1,0,0)$, so w is not the identity transformation. We will prove this by induction on length of w. The base case is trivial; if the length of w is 1, then $w = \sigma$, so $w(1,0,0) = (1, 2\sqrt{2}, 0)/3$. Now, assume that $w = \varpi w'$, where $\varpi \in \{\sigma^{\pm 1}, \tau^{\pm 1}\}$ and the claim holds for w'; that is, $w'(1,0,0) = (a', b'\sqrt{2}, c')/3^{k-1}$. If $\varpi = \sigma^{\pm 1}$, then

$$w(1,0,0) = \sigma^{\pm 1}(a', b'\sqrt{2}, c')/3^{k-1} = \frac{1}{3^{k-1}} \begin{pmatrix} \frac{1}{3} & \mp\frac{2\sqrt{2}}{3} & 0 \\ \pm\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix}$$

$$= \frac{1}{3^{k-1}} \begin{pmatrix} \frac{a' \mp 2b'(\sqrt{2})^2}{3} \\ \frac{\pm 2a'\sqrt{2} + b'\sqrt{2}}{3} \\ c' \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} a' \mp 4b' \\ (b' \pm 2a')\sqrt{2} \\ 3c' \end{pmatrix},$$

satisfying the condition. If $\varpi = \tau^{\pm 1}$, then

$$w(1,0,0) = \tau^{\pm 1}(a', b'\sqrt{2}, c')/3^{k-1} = \frac{1}{3^{k-1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a' \\ b'\sqrt{2} \\ c' \end{pmatrix}$$

$$= \frac{1}{3^{k-1}} \begin{pmatrix} a' \\ \frac{b'\sqrt{2} \mp 2c'\sqrt{2}}{3} \\ \frac{2b'(\sqrt{2})^2 \pm c'}{3} \end{pmatrix} = \frac{1}{3^k} \begin{pmatrix} 3a' \\ (b \mp 2c')\sqrt{2} \\ c' \pm 4b' \end{pmatrix}.$$

This also satisfies the conditions for the induction, which completes the proof. □

Hausdorff's paradox now follows easily.

Corollary 10 (Hausdorff)**.** *There is countable $D \subseteq S^2$ such that $S^2 \setminus D$ is SO_3-paradoxical.*

Proof. Let $F \subseteq SO_3$ be the free group of rotations guaranteed by Theorem 9. Clearly, each rotation in F fixes two points in S^2: the points on the sphere intersecting its axis of rotation. However, let D be the set of all such points. As F is countable, so must D be countable. Hence, F has no nontrivial fixed points on $S^2 \setminus D$, so by Corollary 8, $S^2 \setminus D$ is F-paradoxical, hence SO_3-paradoxical. □

Arguably, most of the work in proving Banach-Tarski lies in proving Hausdorff's paradox. Essentially all that remains to be seen is how to avoid this countable piece.

Theorem 11. *Let $D \subseteq S^2$ be countable. Then, S^2 and $S^2 \setminus D$ are equidecomposable.*

Proof. It suffices to show that there exists a rotation $\rho \in SO_3$ such that $D, \rho(D), \rho^2(D), \ldots$ are pairwise disjoint. Letting $\overline{D} = \bigcup_{n<\omega} \rho^n(D)$, this means $S^2 = \overline{D} \cup (S^2 \setminus \overline{D}) \sim \rho(\overline{D}) \cup (S^2 \setminus \overline{D}) = S^2 \setminus D$, as $\rho(\overline{D}) = \overline{D} \setminus D$. In other words, $\overline{D}$ is the orbit of D under some rotation ρ. Since $\overline{D}$ is obvious equidecomposable with $\rho(\overline{D})$, this allows us to "absorb" the countable set D into S^2.

To find such a rotation ρ, note that SO_3 is uncountable. Hence, as D is countable, we can find an element ρ of SO_3 such that $\rho(D) \cap D = \emptyset$. Hence also $\rho^n(D) \cap D = \emptyset$ for $n > 0$, so $\rho^m(D) \cap \rho^n(D) = \emptyset$, as $\rho^{m-n}(D) \cap D = \emptyset$. That is, this ρ fulfills the above requirement, completing the proof. □

Due to the previously proved relationship between equidecomposability and paradoxical decompositions, if $S^2 \setminus D$ is G-paradoxical, then so is S^2.

Corollary 12. *S^2 is SO_3-paradoxical, as is any sphere centered at the origin.*

Proof. This is immediate from the previous two results; Hausdorff's paradox shows that $S^2 \setminus D$ is paradoxical, for countable D. Theorem 11 then shows S^2 is paradoxical. To get the result for larger spheres, merely note that the proofs of earlier results nowhere depend upon the radius of S^2. That is, any rotational action paradoxical on S^2 is paradoxical on any sphere (centered at the origin), as rotation is clearly independent of the radius of the sphere. □

We are close!

Theorem 13 (Banach-Tarski Paradox)**.** *Any solid ball in $\mathbb{R}^3$ centered at the origin is G_3-paradoxical. Further, any solid ball in $\mathbb{R}^3$ is as well.*

Proof. Let $\mathbf{S} \subseteq \mathbb{R}^3$ be a solid ball centered at the origin. Let the radius of $\mathbf{S}$ be r. Theorem 12 immediately yields that $\mathbf{S} \setminus \{0\}$ (0 denoting the point $(0,0,0)$) is paradoxical, as $\mathbf{S} \setminus \{0\}$ is the union of continuum many spheres—those with radius r' such that $0 < r' \leq r$.[1] Therefore, all that remains to be shown is that $\mathbf{S} \sim \mathbf{S} \setminus \{0\}$. To show this, we use the same trick we used to show that $S^2 \sim S^2 \setminus D$. However, since we are now working in G_3 rather than SO_3, we can use rotations not centered on the origin. Let ρ be such a rotation of infinite order with $\overline{O} = \bigcup_{n<\omega} \rho^n(\{0\}) \subseteq \mathbf{S}$. Then, $\rho(\overline{O}) = \overline{O} \setminus \{0\}$, so $\mathbf{S} \sim \mathbf{S} \setminus \{0\}$.

For solid balls not centered at the origin, merely note that G^3 contains translations; conjugate the transforms for the previous case with the translation corresponding to the offset for the solid ball. □

Corollary 14 (Strong Banach-Tarski Paradox)**.** *Let A and B have nonempty interior and be bounded subsets of $\mathbb{R}^3$. Then, A and B are equidecomposable.*

Proof. We show that $A \preceq B$. By symmetry, the same argument shows that $B \preceq A$, so $A \sim B$, by Theorem 6. Let $\mathbf{S} \supseteq A$ and $\mathbf{T} \subseteq B$ be solid balls. Further, let n be an integer such that $\mathbf{S}$ can be covered by n copies of $\mathbf{T}$. Call the union of this cover C. Now, apply Theorem 13 as necessary to "duplicate" $\mathbf{T}$ and using translations get that $C \preceq \mathbf{T}$. We have $C \preceq \mathbf{T}$ instead of $C \sim \mathbf{T}$ as the copies of $\mathbf{T}$ is C may overlap. But as $A \subseteq \mathbf{S} \subseteq C \preceq \mathbf{T} \subseteq B$, we have $A \preceq B$. □

4. Amenability[2]

The relationship between amenability and paradoxical decompositions is simple; they are mutually exclusive, in a sense that will soon be made formal. First, however, we ought to see a formal definition of amenability.

[1]Like an onion with continuum many layers.

[2]Due to time constraints, the theorems in this section will presented with minimal proof.

Also due to time constraints, this section is much shorter than I wanted it to be. I would have preferred to explore amenability more deeply, but that unfortunately did not happen.

Definition 15. Let G be some group. We say G is *amenable* provided there exists a finitely additive measure μ on $\mathcal{P}(G)$ such that $\mu(G) = 1$ and for all $g \in G$, $A \subseteq G$, $\mu(A) = \mu(gA)$. That is, multiplication in G preserves measure.

The following theorem, due to Tarski, relates this notion of measure with paradoxical decompositions, although the measure is over the set, not the group acting on it.

Theorem 16 (Tarski)**.** *Let G act on X with $Y \subseteq X$. There is a finitely additive, G-invariant measure μ on $\mathcal{P}(X)$ such that $\mu(Y) = 1$ if and only if Y is* not *G-paradoxical.*

Proof. Not given here; see Chapter 9 of [1]. □

Corollary 17. *There is no finitely additive G_3-invariant measure on $\mu(\mathbb{R}^3)$ such that $\mu(\mathbf{S})$ is finite, where $\mathbf{S}$ is the solid unit sphere.*

Proof. As the Banach-Tarski paradox shows, $\mathbf{S}$ is G_3-paradoxical. Hence, there is no finitely additive G_3-invariant measure on $\mu(\mathbb{R}^3)$ with $\mu(\mathbf{S}) = 1$. To show no such measure with $\mu(\mathbf{S})$ finite exists, assume towards contradiction that one does. Call it ν. Then, $\nu(\mathbf{S}) = r \in \mathbb{R}$. But then the measure ν' where $\nu'(A) = \frac{1}{r}\nu(A)$ has $\nu(\mathbf{S}) = 1$, which is a contradiction. □

Of course, Tarski's theorem allows us to directly connect amenability and paradoxicality; the similarity is not coincidental.

Theorem 18. *Let G acting on X be amenable. Then, X is not G-paradoxical.*

Proof. Fix $x \in X$ and let μ a measure on G witnessing G is amenable. Then, define measure ν on X as $\nu(A) = \mu(\{g \in G : g(x) \in A\})$. Clearly, as μ is finitely additive and G-invariant, so is ν. Further, $\mu(X) = 1$, as $\{g \in G : g(x) \in X\} = G$. Hence, by Tarski's theorem, X is not G-paradoxical. □

5. Conclusion

The Banach-Tarski paradox is a deep result which connects many areas of mathematics. Accordingly, it is well beyond the scope of any one paper to address every facet thereof. Nonetheless, there is space to mention (i.e. without proof) one application of Banach-Tarski. Specifically, it can be used to prove the uniqueness of Lebesgue measure. This important connection illustrates why Banach-Tarski is an interesting result. Not only does it increase our understanding of mathematics—by forcing us to confront the difference between proven theorems and naïve conceptions—but it also can be used to formally show further results.

References

[1] Stan Wagon. *The Banach-Tarski Paradox*. Cambridge University Press, 1985.